The Penrose inequality for the perturbed Schwarzschild initial data

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joint work with J. Kopiński

The Penrose inequality

The surface area of the Kerr horizon

$$|S_h| = 8\pi m(m + \sqrt{m^2 - a^2}),$$

hence (the Penrose inequality)

$$m \geq \sqrt{\frac{|S_h|}{16\pi}}\ .$$

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Stronger version

$$E^2 - p^2 \geq \frac{|S_h|}{16\pi} + \frac{4\pi J^2}{|S_h|}\ ,$$

where E, p, J are, respectively, the total energy, momentum and angular momentum.

Consider the Cauchy data with a horizon. If

- a future singularity is surrounded by the event horizon (physically realistic data, the cosmic censorship conjecture)
- configuration tends to a stationary state

then the no-hair theorem etc. (almost) imply that

- the end state is the Kerr metric with E^∞ and S_h^∞
- $E^\infty \leq E$ (because of radiation)
- $|S_h^\infty| \geq |S_h|$ (BH thermodynamics)

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Conclusion: the Penrose inequality should be satisfied on the initial surface

Vacuum initial data with a horizon

Constraints on g'_{ij}, K'_{ij}

$$\nabla_i \left(K'^i_j - H' \delta^i{}_j \right) = 0$$
$$R' + H'^2 - K'^2 = 0$$

where $H' = K'^i_i$ and $K'^2 = K'_{ij} K'^{ij}$ and R' is the Ricci scalar of g'_{ij}.

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Horizon: compact surface with vanishing expansion of outer null rays (marginal outer trapped surface - MOTS)

$$H' - K'_{nn} + h = 0 \ ,$$

where n^i is the unit normal vector and $h = \nabla_i n^i$ is the mean curvature of the surface.

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where n^i is the unit normal vector and $h = \nabla_i n^i$ is the mean curvature of the surface.
If $H' = h = 0$ then the Penrose inequality follows from the Hamiltonian constraint (Geroch, ..., Huisken and Ilmanen).

The conformal approach

$$g'_{ij} = \psi^4 g_{ij}, \quad K'^i_j = \psi^{-6} A^i{}_j + \frac{1}{3} H' \delta^i{}_j \ ,$$

where $A^i{}_i = 0$.

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The Hamiltonian constraint (the Lichnerowicz equation)

$$\triangle \psi = \frac{1}{8} R \psi - \frac{1}{8} A_{ij} A^{ij} \psi^{-7} + \frac{1}{12} H'^2 \psi^5 \ .$$

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The MOTS condition

$$n^i \partial_i \psi + \frac{1}{2} h \psi - \frac{1}{4} A_{nn} \psi^{-3} + \frac{1}{6} H' \psi^3 = 0 \ \ on \ \ S_h \ .$$

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Existence theorems under extra conditions (D. Maxwell, Anglada).

Approximate constraint equations

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and the conformal transformation leads to the Schwarzschild initial metric on $t = const$

$$g' = \frac{dr'^2}{1 - \frac{2m}{r'}} + r'^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

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Let ϵ be a small parameter and

$$\psi = \psi_0 + \psi_1 + \psi_2 + ...$$

$$A^{ij} = B^{ij} + ... \quad H' = B + ...$$

where $\psi_n \sim \epsilon^n$ and $B_{ij}, B \sim \epsilon$.

The approximate constraints

$$\nabla_i B^i{}_j = \frac{2}{3}\psi_0^6 \nabla_j B$$

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- A relation between total energy and area of horizon should follow from the Lichnerowicz equation.
- ψ_1 is harmonic and ψ_2 satisfies the Poisson equation with the Neumann type boundary condition.

Integration of the Lichnerowicz equation

$$\oint_{S_\infty} \psi_{,r} d\sigma = \oint_{S_h} \psi_{,r} d\sigma + \int_{S_h}^{S_\infty} \left(-\frac{1}{8} B_{ij} B^{ij} \psi_0^{-7} + \frac{1}{12} B^2 \psi_0^5\right) d^3x$$

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The ADM total energy

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The ultimate surface area of S_h

$$|S_h| = \oint_{S_h} \psi^4 d\sigma = 16 \oint_{S_h} (1 + 2\psi_1 + 2\psi_2 + \frac{3}{2}\psi_1^2) d\sigma \ .$$

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Integration of the Lichnerowicz equation over spherical coordinates

$$\frac{1}{r^2} \partial_r (r^2 \partial_r \langle \psi \rangle) = \langle -\frac{1}{8} B_{ij} B^{ij} \psi_0^{-7} + \frac{1}{12} B^2 \psi_0^5 \rangle$$

where $\langle f \rangle = \int \int f \sin\theta d\theta d\phi$. An expression for $|S_h|$ follows.

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$$+\frac{m}{\pi}\int_{\frac{m}{2}}^{\infty} r^2(1-\frac{m}{2r})\langle \frac{1}{8}B_{ij}B^{ij}\psi_0^{-7} - \frac{1}{12}B^2\psi_0^5\rangle dr \ ,$$

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Arbitrary sign of P_l for unrestricted fields B_{ij}, B.

Axially symmetric perturbations

One of the momentum constraints:

$$B_{\varphi\theta} = \frac{\omega_{,r}}{\sin\theta}\ , \quad B_{\varphi r} = -\frac{\omega_{,\theta}}{r^2\sin\theta}\ .$$

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Contribution to P_I

$$P_J = 2\int_{\frac{m}{2}}^{\infty} dr\,\varrho r \int_{-1}^{1} (B_{\varphi\theta}^2 + r^2 B_{\varphi r}^2)\frac{dz}{1-z^2}\ ,$$

where $z = \cos\theta$ and

$$\varrho = \frac{m(1-\frac{m}{2r})}{4r^3\psi_0^7}\ .$$

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From regularity conditions

$$\omega = f\sin^4\theta + J(\cos^3\theta - 3\cos\theta) \ , \quad J = ang.mom.$$

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Simple consequence

$$P_J \geq \frac{J^2}{4m^2} \Rightarrow P_J \geq \frac{4\pi}{|S_h|} J^2 \;.$$

Other 2 momentum constraints (for 4 functions)

$$(r^3 B_{rr} \sin\theta)_{,r} + (r B_{r\theta} \sin\theta)_{,\theta} = \frac{2}{3} r^3 \psi_0^6 B_{,r} \sin\theta \ ,$$

$$(B_{\theta\theta} \sin^2\theta)_{,\theta} + (r^2 B_{r\theta})_{,r} \sin^2\theta + r^2 B_{rr} \sin\theta \cos\theta = \frac{2}{3} r^2 \psi_0^6 B_{,\theta} \sin^2\theta \ .$$

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Contribution to the volume integral in P_I

$$\tilde{P}_I = \int_{\frac{m}{2}}^{\infty} dr \varrho r \int_{-1}^{1} [2r^2 B_{r\theta}^2 + 2(B_{\theta\theta} + \frac{1}{2} r^2 B_{rr})^2 + \frac{3}{2} r^4 B_{rr}^2 - \frac{2}{3} r^4 \psi_0^{12} B^2] dz$$

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$$= \frac{m}{2} \int_{\frac{m}{2}}^{\infty} \frac{dr}{r^2 \psi_0} \int_{-1}^{1} ((1 - \frac{m}{2r}) W^2 - X \triangle_s Y) dz + \frac{1}{4} \int_{-1}^{1} X_h^2 dz \ ,$$

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where

$$\partial_z\left((1 - z^2) W\right) = \frac{1}{1 - \frac{m}{2r}} (z^2 - 1)\partial_z (rX_{,r} + \frac{1}{2}\psi_0 \triangle_s Y + (\psi_0 - \frac{3m}{r\psi_0}) Y) \ .$$

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$$W = \Sigma W_n(X_n, Y_n)\tilde{P}_n \ , \ n \geq 2 \ ,$$

where polynomials

$$\tilde{P}_n = \frac{1}{n+2}(nP_n - \frac{2}{n-1}P_{n-1,z})$$

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Each F_n has global minimum at $Y_n = \alpha_n X_{n,r} + \beta_n X_n$.

Crucial estimation

$$\tilde{P}_l \geq p^2 + \frac{1}{2}\Sigma\frac{X_{nh}^2}{(2n+1)(n^2+n+1)}$$

Here X_{nh} are multipole moments of the boundary term $(\frac{1}{32}B_{rr} - \frac{4}{3}B)_h$.

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$$\tilde{P}_I \geq p^2 + \frac{1}{2}\Sigma\frac{X_{nh}^2}{(2n+1)(n^2+n+1)}$$

Here X_{nh} are multipole moments of the boundary term $(\frac{1}{32}B_{rr} - \frac{4}{3}B)_h$. Taking into account inequalities for P_J, $\tilde{P}_I$ and boundary terms in P_I yields

$$E^2 - \frac{|S_h|}{16\pi} \geq \frac{4\pi}{|S_h|}J^2 + p^2 + \Sigma_2^\infty\frac{(n-1)(n+2)X_{nh}^2}{2(2n+1)^3(n^2+n+1)} \ .$$

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If any of X_{nh} with $n \geq 2$ is different from zero (generic case) then the Penrose inequality is satisfied.
If $X_{n\geq 2} = 0$ (nongeneric case) then higher order terms decide about validitity of the Penrose inequality.

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- initial exterior curvature is proportional to a small parameter ϵ and all fields can be expanded in ϵ

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Perspectives: dependence on ϕ, nonspherical MOTS, ...