The asymptotic analysis of the lorentzian KKL vertex of arbitrary valence

## Simone Speziale JurekFest Warsaw 18 september 2019

Based on

P. Donà, M. Fanizza, G. Sarno and SiS, *SU(2) graph invariants, Regge actions and polytopes* (1708.01727) and on work with P. Donà to appear



## Aim of the talk

Offer a nice formula as a gift to Jurek for his 60th birthday!

- There exist a covariant framework for the dynamics of LQG, known as <u>spin foam formalism</u>
- This is particularly developed in the case of **4-valent spin networks**, which are dual to 3d simplicial triangulations and presents a simple interpretation in terms of **discrete** geometries
- For these, the spin foam amplitude for one 4-simplex is dominated in the semiclassical limit by exponentials of the **Regge action**
- Quite a nice state of affair, even though many open questions remains (notably on the semiclassical limit of an extended triangulation and curved solutions)
- Another question is the limitation to 4-valent spin networks, which is not very democratic since the LQG Hilbert space contains a priori states of any valency



## Jurek's role



Jurek and collaborators answered this question setting the EPRL model on broader and firmer grounds and extending its validity:

Kaminski-Kisielowski-Lewandowski: Spin-Foams for All Loop Quantum Gravity 0909.0939

Ding-Han-Rovelli: Generalized Spinfoams 1011.2149, (see also Baratin-Flori-Thiemann '08)

#### Ok so we have a generalized vertex. But how about its semi-classical limit?

The EPRL 4-simplex amplitude is dominated by Regge configurations

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Suppose the vertex graph is dual to the boundary of a polytope;

- Are we going to get a Regge action for a flat polytope, as opposed to a flat 4-simplex?
- Or since a polytope can be chopped into 4-simplices, maybe one gets a Regge action for a curved polytope?
- Or something else?

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Turns out to be dominated by configurations which are more general than Regge's, which we called conformally matched twisted geometries, and the amplitude is well approximated by an action which resembles the Regge action but has different equations of motion (first observed by Bahr and Steinhaus in two different examples, our Marseille contribution is to have given a complete analysis)

• Regge configurations are only a subset of the dominant ones

## Outline

- Preliminaries
- 4-simplex asymptotics revisited
- Arbitrary vertex: SU(2) case
- Arbitrary vertex: Lorentzian KKL case
- Conclusions and perspectives

## I. Preliminaries

## Quanta of space in loop quantum gravity

R R Š V, p Quantum field theory

Loop quantum gravity

$$\mathcal{F} = \bigoplus_n \mathcal{H}_n$$

 $|n, p_i, h_i 
angle 
ightarrow$  quanta of fields

- number of quanta
- momenta
- helicites

Fuzzy spinning particles

dynamics: described by Feynman diagrams



diagrams can be organised in PT or EFT

$$\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_{\Gamma}$$



as abstract graphs;

inductive limit

for embedded graphs

 $|\Gamma, j_e, i_v 
angle 
ightarrow$  quanta of space

- number of quanta and their relations
- volumes of regions
- areas of interconnecting surfaces

Distributional or fuzzy polyhedra interpretation

**dynamics:** Hamiltonian approach or described by **spin foams** 



*diagram organisation not yet established!* hands-on approach for the moment: compute in a given **truncation**, then change truncation

## Semiclassical limit at fixed graph

In the large spin limit



## **Brief reminder of Regge calculus**

$$M \longrightarrow \Delta \qquad g_{\mu\nu} \longrightarrow \{l_e\}$$



 $\epsilon_e^{3d} = 2\pi - \sum_{\tau \ni e} \varphi_e^{\tau}(l_e)$  dihedral angles between two **triangles** in a given tetrahedron  $\tau$ 

 $\epsilon_t^{4d} = 2\pi - \sum \theta_t^{\sigma}(l_e) \longleftarrow$  dihedral angles between two **tetrahedra** in a given tetrahedron  $\sigma$ 

## 4d Regge action

$$S_{\rm R}(l_e) = \sum_t j_t(l_e)\epsilon_t(l_e)$$

• can be written also as a sum of **simplicial terms**:

$$S_{\sigma}(l_e) = \sum_t j_t(l_e)\theta_t(l_e)$$

• can be written also with areas and (3d dihedral) angles as fundamental variables:

$$S_{AA}(j_t, \varphi_e^{\tau}) = \sum_t j_t \epsilon_t(\varphi) + \sum_{\tau} \lambda_{\tau} C_{\tau}(j, \varphi) + \sum_{ee'} \mu_{ee'} S_{ee'}(\varphi) \qquad \text{Dittrich-SiS '08}$$

$$closure \\ conditions \qquad conditions \qquad conditions \qquad conditions \qquad \text{orditions} \qquad \text{Dittrich-SiS '08}$$
whose simplicial term is 
$$S_{\sigma}(j_t, \varphi_e^{\tau}) = \sum_{\tau} j_t \theta_t(\varphi) + \text{constraints}$$

## **Simplex rigidities**

There is an obvious reason why Regge chose to work with simplices:

- Described by the complete graph in any dimensions, trivial adjacency matrix
- Their geometry in Euclidean space uniquely determined by the edge lengths





Generic polytopes do not share this edge-length rigidity











Edge lengths too many!

⇒ Flat-embeddability conditions

## **Simplex rigidities**

Generic polytopes do not share this edge-length rigidity



4-simplex	$\left(5,20,10,5\right)$	$\{3\},\{3,3\}$	Υ	10
tesseract	(16, 32, 24, 8)	$\{4\}, \{4,3\}$	Υ	22
orthoplex	(8, 24, 32, 16)	$\{3\},\{3,3\}$	Т	46
octoplex	(24, 96, 96, 24)	$\{3\},\{3,4\}$	Ι	?
duo-prisms :	(16, 32, 24, 8)		Υ	22
	(64, 128, 80, 16)		Υ	54
	$\left(9,18,15,6\right)$	$9{4} + 6{3}$	Υ	14

II. SU(2) BF theory asymptotics and polytopes

## 3d example: the SU(2) 6j asymptotics

(Wigner, Ponzano-Regge, Schouten-Gordon...)

$$\{6j\} \sim \frac{1}{j^{3/2}} \cos\left(\sum_{e} j_e \phi_e(j) + \frac{\pi}{4}\right) \qquad V^2(j) > 0$$
3d Regge action for a tetrahedron

and one can similarly study such homogeneous large spin limits for higher order {nj}-symbols

e.g. the {9*j*} studied by Haggard and collaborators





- all spin and intertwiner large: some 3d interpretation
- spins large at fixed intertwiners: 4d interpretation!

4d case more interesting due to non-commutativity of the geometry

#### $V_{i_3}$ $V_{i_4}$ SU(24) 4-simplex vertex amplitude

The large-spin/fixed-interwiner limit is easier to study if one uses coherent intertwiners

$$\sum_{m_i} \left( \begin{array}{ccc} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{array} \right)^{(j_{12})} \langle j_i, m_i | j_i, \vec{n}_i \rangle = \overbrace{j_1}^{n_2} \overbrace{j_2}^{j_2} \overbrace{j_3}^{j_3} \overbrace{j_4}^{n_4}$$

## SU(2) BF theory:

• 4-simplex vertex amplitude

$$\vec{n}_{1} \quad \vec{n}_{2} \quad \vec{n}_{3} \quad \vec{n}_{4} \quad i_{2} \quad j_{13} \quad j_{14} \quad j_{15} \quad i_{5} \quad i_{14} \quad j_{15} \quad j_{14} \quad j_{15} \quad j_{15} \quad j_{14} \quad j_{15} \quad j_{16} \quad j_{1$$

• 4-simplex coherent vertex amplitude

$$A_{v}(j_{ab},\vec{n}_{ab}) = \int \prod_{a} dg_{a} \prod_{(ab)} \langle -\vec{n}_{ab} | g_{a}^{-1} g_{b} | \vec{n}_{ba} \rangle^{2j_{ab}} = \sum_{\{i_{a}\}} \prod_{a} d_{i_{a}} \int_{j_{23}}^{j_{23}} \int_{j_{24}}^{j_{24}} \int_{j_{35}}^{j_{14}} \int_{j_{45}}^{j_{14}} \int_{j_{45}}^{j_{45}} \int_{j_{45$$

**Remark:** boundary data are purely 3d, but every 3d Regge geometry of 5 tetrahedra patched together is the boundary of a flat 4-simplex

This is the simple reason why SU(2) - rotations in 3d - can be used to construct a 4d object





SU(2) 4-simplex asymptotics reviewed

$$A_v(j_{ab}, \vec{n}_{ab}) \sim \frac{1}{j^6} \cos\left(\sum_t j_t \theta_t(j)\right)$$
 (Barrett et al. '09)

**Critical point equations:** 

**1.**  $C_a = \sum_{b \neq a} j_{ab} \vec{n}_{ab} = 0$  **Closure conditions:** existence of a tetrahedron at each node

**2.**  $R_a \vec{n}_{ab} = -R_b \vec{n}_{ba}$  **Orientation conditions:** existence of rotations making the normals pairwise anti-parallel

Data satisfying both conditions define what Barrett called vector geometries

• For generic boundary data these conditions are not satisfied → **exponential fall-off** 



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- **Special Regge subset** admits *two* distinct critical points → same **power law** with invariant oscillations



of boundary data:

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## SU(2) 4-simplex asymptotics revisited

$$A_v(j_{ab}, \vec{n}_{ab}) \sim \frac{1}{j^6} \cos\left(\sum_t j_t \theta_t(j)\right)$$

dim.	geometry type	saddles
20	twisted	0
15	vector (anti-parallel)	1
10	Regge (angle-matching)	2

**Critical point equations:**  $C_a = \sum_{b \neq a} j_{ab} \vec{n}_{ab} = 0$   $R_a \vec{n}_{ab} = -R_b \vec{n}_{ba}$ 

## **Standard procedure (**Barrett and collaborators):

1. define bivectors from the 3d vectors and the SU(2) group elements at the critical point

2. apply bivector reconstruction theorem to classify critical point solutions

(+): rigorous

(-) : applies only to this 4-simplex vertex amplitude, obscures some of the geometry (at least to me)



## SU(2) 4-simplex asymptotics revisited

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Alternative procedure (Donà): fix a convenient gauge, reconstruct the 3d geometry, study the flatembedding of the 3d geometry

1. Use gauge freedom to fix the **twisted spike** gauge  $\vec{n}_{ab} = -\vec{n}_{ba}$   $\Rightarrow R_a = \mathbb{I}$  trivially solution To find other sols:

2. Pick 1 as base node, and explicitly solve using Rodrigues' formula  $\Rightarrow R_a := e^{2i\theta_{1a}\vec{n}_{a1}\cdot\vec{J}}$ 

$$\vec{n}_{ab} \cdot \vec{n}_{ac} = \cos \varphi_{bc}^{(a)} \qquad \cos \theta_{1a} = \frac{\cos \varphi_{1a}^{(b)} + \cos \varphi_{1b}^{(a)} \cos \varphi_{ab}^{(1)}}{\sin \varphi_{1b}^{(a)} \sin \varphi_{ab}^{(1)}} \checkmark$$

## Existence of the second solution requires the shape-matching conditions

⇒ connection to area-Regge calculus and secondary simplicity constraints clarified

To be more precise angle-matching conditions, but equivalent for triangles since areas match already

## The spike and the twisted spike

Normals pairwise opposite describe a unique Euclidean 4-simplex

 $\vec{n}_{ab} = -\vec{n}_{ba}$ 

Start from a Euclidean 4-simplex

Pick a reference tetrahedron, SO(4)-rotate the other 4 tetrahedra to lie in this hyperplane

**Spike** configuration: a 3d rendering of the 4-simplex

> Twist the 4 tetrahedra by **exactly** (twice) the 4d difference

> > **Twisted spike** configuration: a 3d rendering of the 4-simplex with extrinsic geometry encoded in the twist

Four pairs of normals are opposite, but not the rest

## SU(2) general vertex asymptotics

#### The lessons from revisiting the 4-simplex asymptotics:

## • existence of a second critical point requires angle-matching conditions

if faces are all triangles this is equivalent to shape-matching  $\rightarrow$  a Regge geometry

• the critical data really determine a 3d Regge triangulation

but all Regge triangulations of 5 tetrahedra glued together are 1-to-1 with flat 4-simplices

#### For more general graphs we can iterate the procedure:





(Remark: specifics of the graph become important! For our purposes, we focus on graphs dual to boundary of polytopes)

- existence of a second critical point requires angle-matching conditions if faces are not triangles we have only a conformal matching
- the critical data do not determine a 3d Regge geometry in general



## More than two critical points

#### Whether this occurs depends on the connectivity of the graph.

the multiplicity of critical points comes from sign options in solving the spherical cosine laws for the angles; this sign freedom is fixed up to a global sign in the 4-simplex and in many other graphs like for example:



 $\rightarrow$  one cosine

The **fixing** uses consistency conditions at the *triangular cycles* of the graph It doesn't work for instance for this graph:



 $\rightarrow$  two cosines

The technical reason being that there are no triangular cycles including the `bridging' links

## **Classification of boundary data**

4-simplex graph

polytope graph

dim.	geometry type	saddles	dim.	geometry type	saddles
20	twisted	0	5L-6N	twisted	0
15	vector (anti-parallel)	1	3L-3N	vector (anti-parallel)	1
10	Regge (angle-matching)	2		conformal twisted (angle-matching)	$\geq 2$

For all conformal twisted data, the asymptotics takes a cosine form with a Regge-looking action:

$$A_v^{\rm LO}(j_{ab}, \vec{n}_{ab}) \sim \lambda^{-\frac{3}{2}(N-1)} \cos\left(\sum_{ab \ 1st \ n.} j_{ab} \theta_{ab}(\varphi)\right)$$

action is defined with closure and angle-matching conditions, and it is **not** equivalent to the action for Regge calculus: there are no well-defined edge lengths in general 4d dihedral angles in terms

*Remark:* shape matchings not needed to define  $\theta_{ab}(\varphi)$ 

.4d dihedral angles in terms of 3d dihedral angles via spherical cosine laws

## **Regge sub-cases**

polytope graph

dim.	geometry type	saddles
5L - 6N	twisted	0
3L - 3N	vector (anti-parallel)	1
	conformal twisted (angle-matching)	$\geq 2$

Among the configurations with (at least) two critical points there are **two interesting sub-cases**:

shape-matched data: not only the angles match, but the edge-lengths
 these configurations describe 3d Regge geometries

A 3d triangulation can not always be flat-embedded The shape-matched data do not always describe the boundary of flat polytopes; they describe in general the boundary of **curved** polytopes

• **flat polytope data:** data that can be flatly embedded

 $A_v^{\rm LO}(j_{ab}, \vec{n}_{ab}) \sim \lambda^{-\frac{3}{2}(N-1)} \cos\left(\sum_{ab \ 1st \ n.} j_{ab} \theta_{ab}(\varphi)\right)$ 

Non-trivial mathematical problem how to characterize explicitly these conditions see proposal in 1708.01727

## **Classification of boundary data**

## 4-simplex graph

#### polytope graph

dim.	geometry type	saddles
20	twisted	0
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dim.	geometry type	saddles
5L-6N	twisted	0
3L-3N	vector (anti-parallel)	1
	conformal twisted (angle-matching)	$\geq 2$
$\left  2L - 2N \right $	Regge (shape-matching)	$\geq 2$
4N - 10	polytope (flat embedding)	$\geq 2$

$$A_v^{\rm LO}(j_{ab}, \vec{n}_{ab}) \sim \lambda^{-\frac{3}{2}(N-1)} \cos\left(\sum_{ab \ 1st \ n.} j_{ab} \theta_{ab}(\varphi)\right)$$

## **III. Lorentzian KKL asymptotics**

P. Donà-SiS to appear

#### EPRL 4-simplex vertex amplitude

$$A_v(j_{ab}, \vec{n}_{ab}) := \int \prod_{a=2}^5 dh_a \prod_{a < b} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{ba}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{ba}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{ba}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{ba}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{ba}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{bb}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{bb}}^{(\gamma j_{ab}, j_{ab})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{bb}}^{(\gamma j_{ab}, j_{ab}, \vec{n}_{bb})} (h_a^{-1} h_b) = \sum_{l_{ab}, k_a} D_{j_{ab}, -\vec{n}_{ab}, j_{ab}, \vec{n}_{bb}}^{(\gamma j_{ab}, j_{ab}, \vec{n}_{bb}, \vec{n}_{$$

#### Differences and similarities with the SU(2) BF case:

- Non-compact group SL(2,C)
- Same graph combinatorics

#### **Very similar asymptotic behaviour** (Barrett et al. '09)



Can be recasted as a convolution of SU(2) 15j symbols (SiS '17)



⇒ The simplicity constraints imposed in the EPRL model play no role in selecting the boundary data but they do in extracting the Regge action, at least at the level of a single 4-simplex

## **EPRL 4-simplex asymptotics reviewed**

$$A_v^{\text{EPRL}}(j_{ab}, \vec{n}_{ab}) \sim \frac{1}{j^{12}} \cos\left(\gamma \sum_t j_t \theta_t^{\text{L}}(j)\right)$$
 (Barrett et al. '09)

Critical point equations: 1.

1. 
$$C_a = \sum_{b \neq a} j_{ab} \vec{n}_{ab} = 0$$
  
2.  $D^{(1)}(h_a) \vec{n}_{ab} = -D^{(1)}(h_b) \vec{n}_{ba}$ 



15

 $j_{14}$ 

35

 $l_{25}$ 

 $l_{34}$ 

Ĵ13/

 $\mathcal{J}24$ 

 $\sum_{l_{ab},k_a,i_a}$ 

 $j_{23}$ 

 $j_{23}$ ° $j_{13}$ 

 $j_{35}$   $j_{2}$ 

 $j_{15}$ 

 $j_{45}$ 

 $\langle j_{45} \\ j_{14} \rangle$ 

 $j_{34}^* j_{24}^{j_{24}}$ 

~  $j_{25}$ 

 $J_{35}$ 



- 2.  $D^{(1)}(h_a)\vec{n}_{ab} = -D^{(1)}(h_b)\vec{n}_{ba}$
- If the vectors are such that the second conditions can be solved with  $h \in SU(2)$  $\Rightarrow$  critical points of SU(2) theory, vector and Euclidean Regge geometries
- If the vectors are such that it is solved with h ∈ SL(2, C) with non-vanishing boosts,
   ⇒ new critical configurations, that can be shown correspond to Lorentzian 4-simplices with all triangles space-like

$$D^{(1)}(h_a)\vec{n}_{ab} = -D^{(1)}(h_b)\vec{n}_{ba}$$

$$h_a \notin \mathrm{SU}(2)$$



$$R_a \vec{n}_{ab} = -R_b \vec{n}_{ba}$$

## **EPRL 4-simplex asymptotics revisited**



As before, we revisite the analysis starting from data  $\vec{n}_{ab} = -\vec{n}_{ba}$  with normals pairwise-opposite and second solution from spherical cosine laws with angle-matching conditions satisfied

$$h_a = \exp\{2i\theta_{a1}^{\mathrm{E}}(\varphi)\vec{n}_{a1}\cdot\vec{J}\} \in \mathrm{SU}(2)$$

## **EPRL 4-simplex asymptotics revisited**



#### No twisted spike data; simplest choice are the spike data

(always accessible - WIP)



As before, we revisite the analysis starting from data  $\vec{n}_{ab} = -\vec{n}_{ba}$  with normals pairwise-opposite and second solution from spherical cosine laws ith angle-matching conditions satisfied

$$h_a = \exp\{2i\theta_{a1}^{\mathrm{E}}(\varphi)\vec{n}_{a1}\cdot\vec{J}\} \in \mathrm{SU}(2)$$

 $\checkmark$  SL $(2, \mathbb{C})$ 

Using a **complex** version of Rodrigues' formula, find sols  $h_a = \exp\{(\pm \theta_{a1}^{L}(\varphi) + i\pi)\vec{n}_{a1} \cdot \vec{J}\}$ with  $\theta$  given in function of the 3d normals' angles  $\varphi$  via Lorentzian spherical cosine laws up to a sign  $\rightarrow$  always 0 or 2 solutions

## **EPRL general vertex asymptotics**

The procedure spike-data+Rodrigues and spherical cosine laws can be extended to general vertices Again, we focus on graphs dual to boundaries of polytopes (Limited additional technical difficulties wrt SU(2) case include use of spinors, infinite-dim irreps and time-reversal maps to make 4-normals all outgoing)

• Characterization of boundary data and corresponding # of critical points like in SU(2) BF

#### plus the novelty of Lorentzian data



• Leading order of asymptotic formula scales with the number of nodes, and the 4d dihedral angles are defined through the spherical cosine laws

$$A_v^{\rm LO}(j_{ab}, \vec{n}_{ab}) \sim \lambda^{-3(N-1)} \cos\left(\gamma \sum_{ab \ 1st \ n.} j_{ab} \theta_{ab}^{\rm L}(\varphi)\right)$$

exact phase and Hessian also computed

## Implications

- The asymptotic analysis of the 4-simplex vertex can be generalized to any vertex graph
- For vertex graphs dual to the boundary of polytopes, one gets an interesting classification of critical points in terms of the geometry of boundary data
- In general critical behaviour also for non-Regge data: conformal twisted geometries
- Action at the critical point well-defined and Regge-looking, but **inequivalent**
- Regge data and flat polytopes are subsets of the set of critical points

# Open question is the physics of this more general dynamics, or whether one should modify the KKL model to have critical behaviour for Regge data alone

You may argue that the critical behaviour of non-Regge data is due to lack of `volume simplicity constraints': absent in the EPRL/KKL construction, the bivector reconstruction theorem proves that these are not needed but works only for a 4-simplex

- (Bahr, Belov are exploring ideas along these lines)
- However we have seen that the situation is similar between SU(2) BF and KKL
- So any such modification would probably modify the 4-simplex amplitude as well
- Which could be good: maybe remove vector geometries from critical behaviour...
- (see also Engle, Zipfel, Han, ...)

**IV.** Numerical tests for the 4-simplex amplitudes

## SU(2) 4-simplex asymptotic formula

Donà-Fanizza-Sarno-SiS 1708.01727 Donà-Sarno 1807.03066 Donà-Fanizza-Sarno-SiS 1903.12624

$$A_v(j_{ab}, \vec{n}_{ab}) \sim \frac{1}{j^6} \cos\left(\sum_t j_t \theta_t(j)\right)$$



## EPRL 4-simplex asymptotic formula, Euclidean data

Donà-Fanizza-Sarno-SiS 1903.12624

$$A_v^{\rm EPRL}(j_{ab}, \vec{n}_{ab}) \sim j^{-12} \cos\left(\sum_t j_t \theta_t^{\rm E}(j)\right)$$



#### EPRL 4-simplex asymptotic formula, Lorentzian data

Donà-Fanizza-Sarno-SiS 1903.12624

$$A_v^{\text{EPRL}}(j_{ab}, \vec{n}_{ab}) \sim j^{-12} \cos\left(\gamma \sum_t j_t \theta_t^{\text{L}}(j)\right)$$



Power law and  $\gamma$ -dependence of frequency confirmed; better matching requires more numerical power! Or better methods...

## Conclusions

Vertex definitions and asymptotic formulas:

```
'07/08 EPR, FK, LS, EPRL 4-simplex vertex amplitude
'09 4-simplex asymptotics (Barrett)
'10 KKL, RY generalized vertex
'15/16 Bahr, Steinhaus first evidence of non-Regge cosines
'18 Donà-Fanizza-Sarno-SiS SU(2) analysis
'19 Donà-SiS Asymptotics of Lorentzian generalized vertex (to appear)
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We know have a pretty complete idea of the asymptotics of the Lorentzian KKL model for nonsimplicial vertices: the fundamental and simple observation is to go beyond bi-vectors, and put to the forefront the role of spherical cosine laws and the fact that the quantity  $\sum j_t \theta_t(\varphi)$  is well-defined for all distinct critical points, regardless of whether they describe Regge geometries or not

#### Conclusions





# Happy birthday Jurek!

Beijing '09