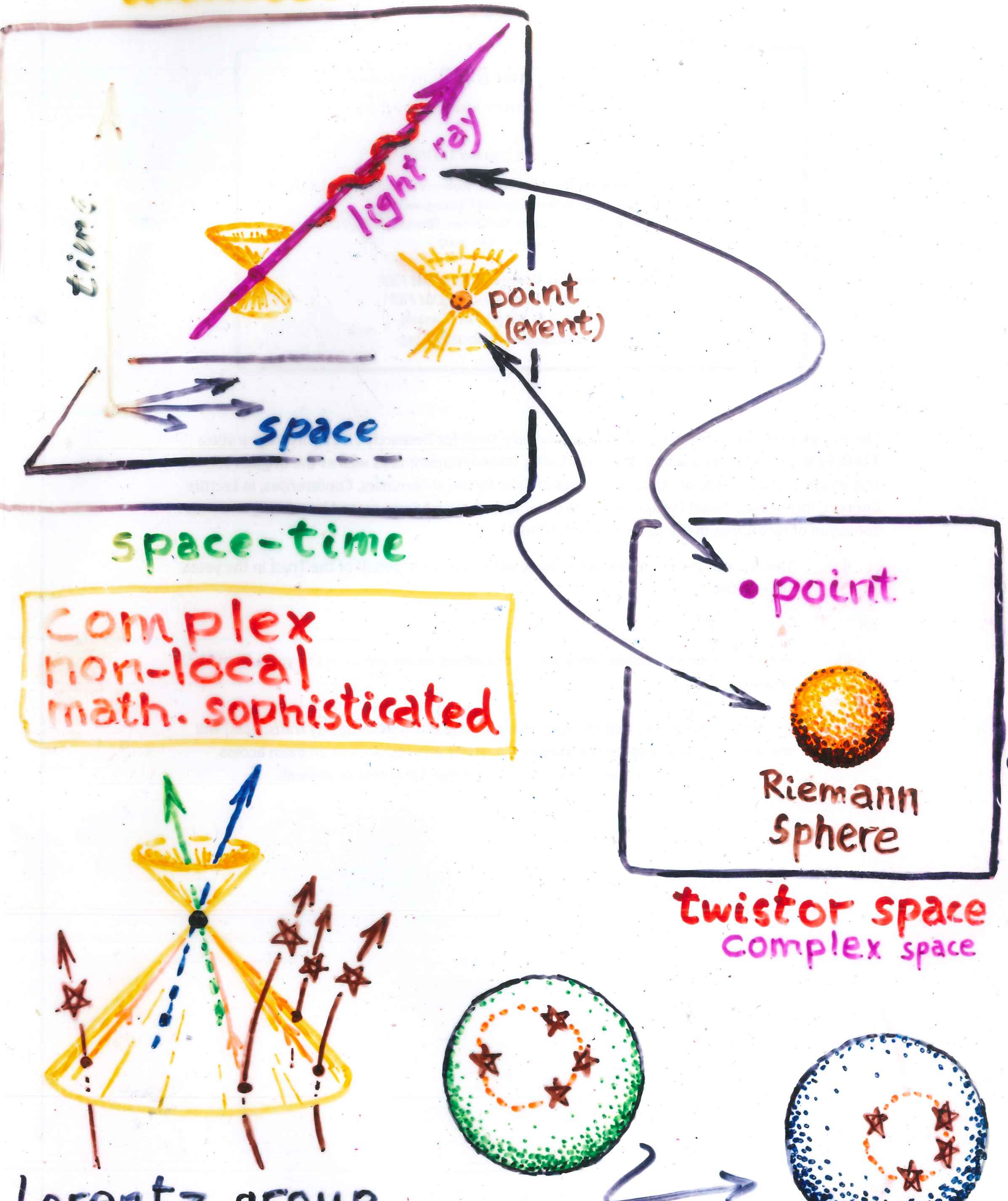
Twistor Theory

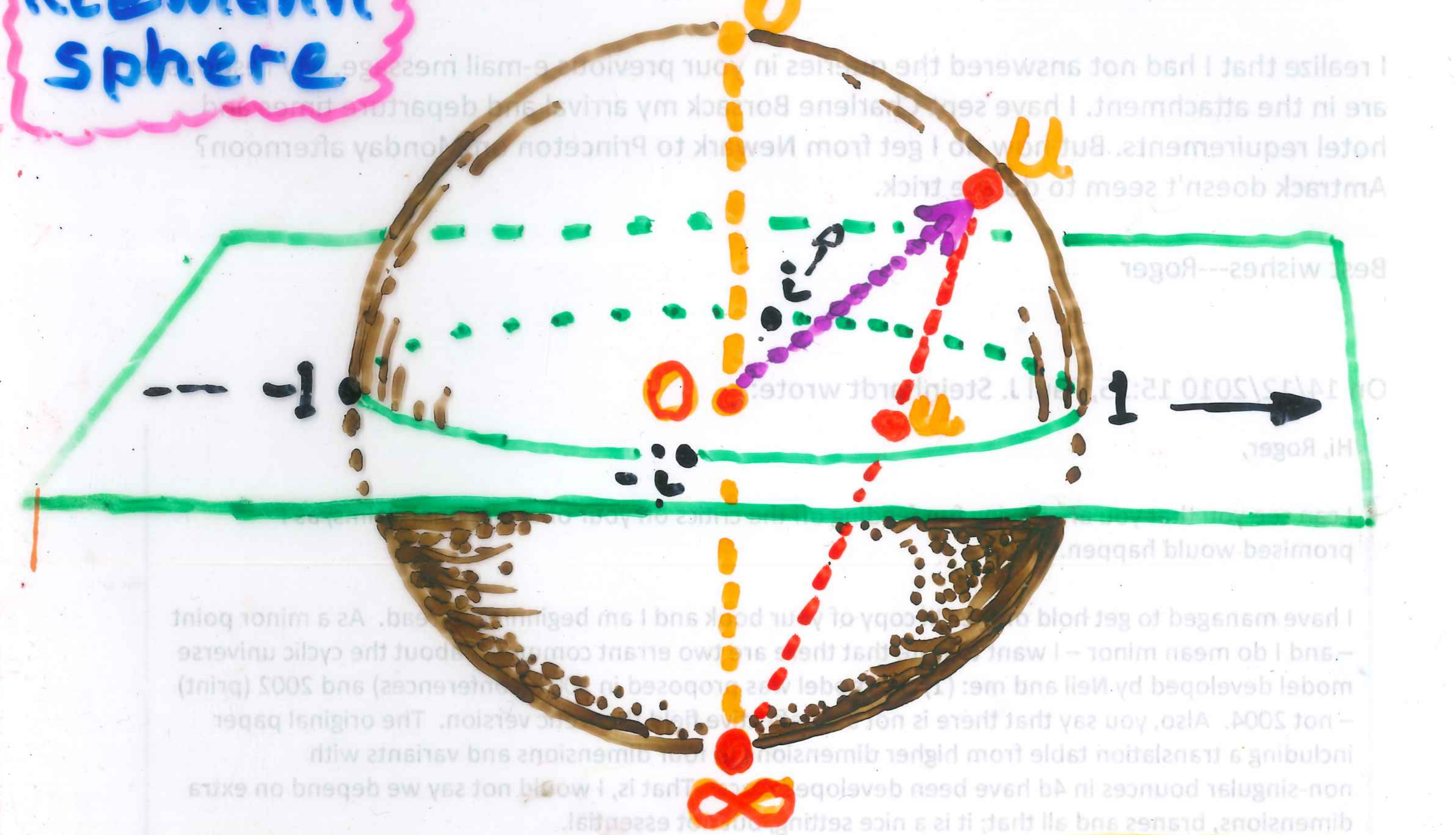


Lorentz group

celestial spheres are Riemann spheres

Stereographic projection

Riemann sphere



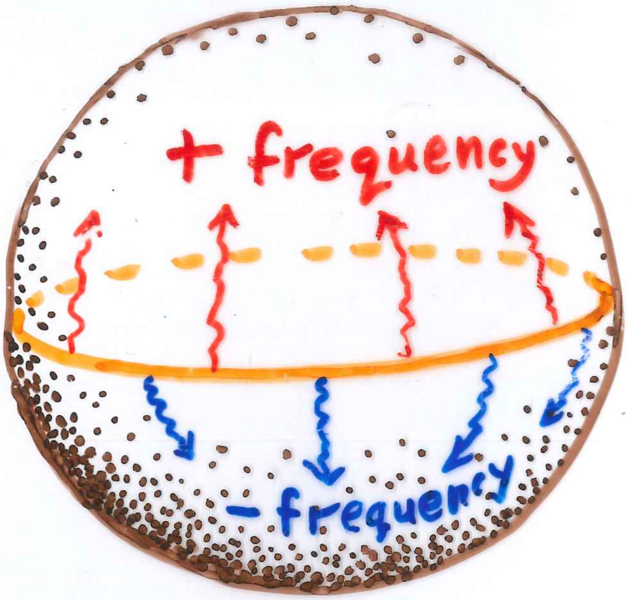
Spin $\frac{1}{2}$ particle (e.g. electron, proton, neutron, quark)

$$u = \frac{z}{w}$$

Riemann sphere

Positive/Negative Frequency Splitting

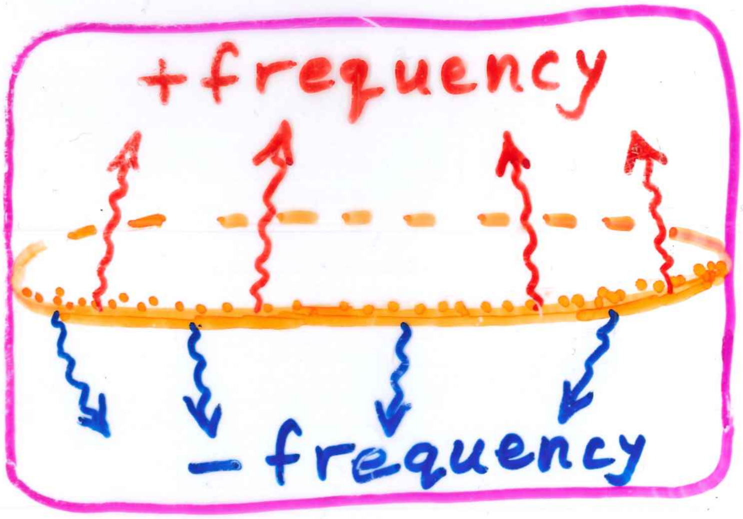
Riemann sphere $\mathbb{CP}^1$



Splitting of H^0's
(ordinary functions)
by holomorphic extension

real axis

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$



splitting of H^1's, by
holomorphic extension

$\mathbb{PN}$
(null twistors)

This realized (finally!) one of the very early motivations behind twistor theory

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} r^0+r^3 & r^1+ir^2 \\ r^1-ir^2 & r^0-r^3 \end{bmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

$$\omega^A = i \, r^{AA'} \, \pi_{A'}$$

$$Z^\alpha = (\omega^A, \pi_{A'})$$

$$\boxed{\text{incidence: } \omega = i\, r \cdot \pi}$$

Shift of origin

$$\overset{Q}{\omega}{}^A = \overset{O}{\omega}{}^A - i\, q^{AA'} \overset{O}{\pi}_{A'}$$

$$\overset{Q}{\pi}_{A'} = \overset{O}{\pi}_{A'}$$

complex conjugate twistor

$$\boxed{\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})}$$

Null twistor

$$Z^\alpha \bar{Z}_\alpha = 0$$

← equation of $\mathbb{P}N$

Momentum-angular mom. for massless ptcle.

p_a = 4-momentum $\quad p_a p^a = 0, \; p_0 > 0$

M^{ab} = 6-angular momentum

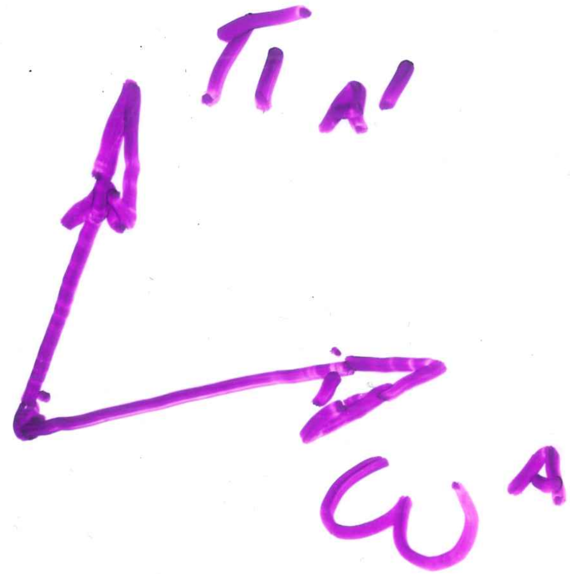
Pauli-Lubanski spin vector → $S_a = \frac{1}{2} e_{abcd} \underbrace{p^b M^{cd}}_{*(p\wedge M)} = s\, p_a$

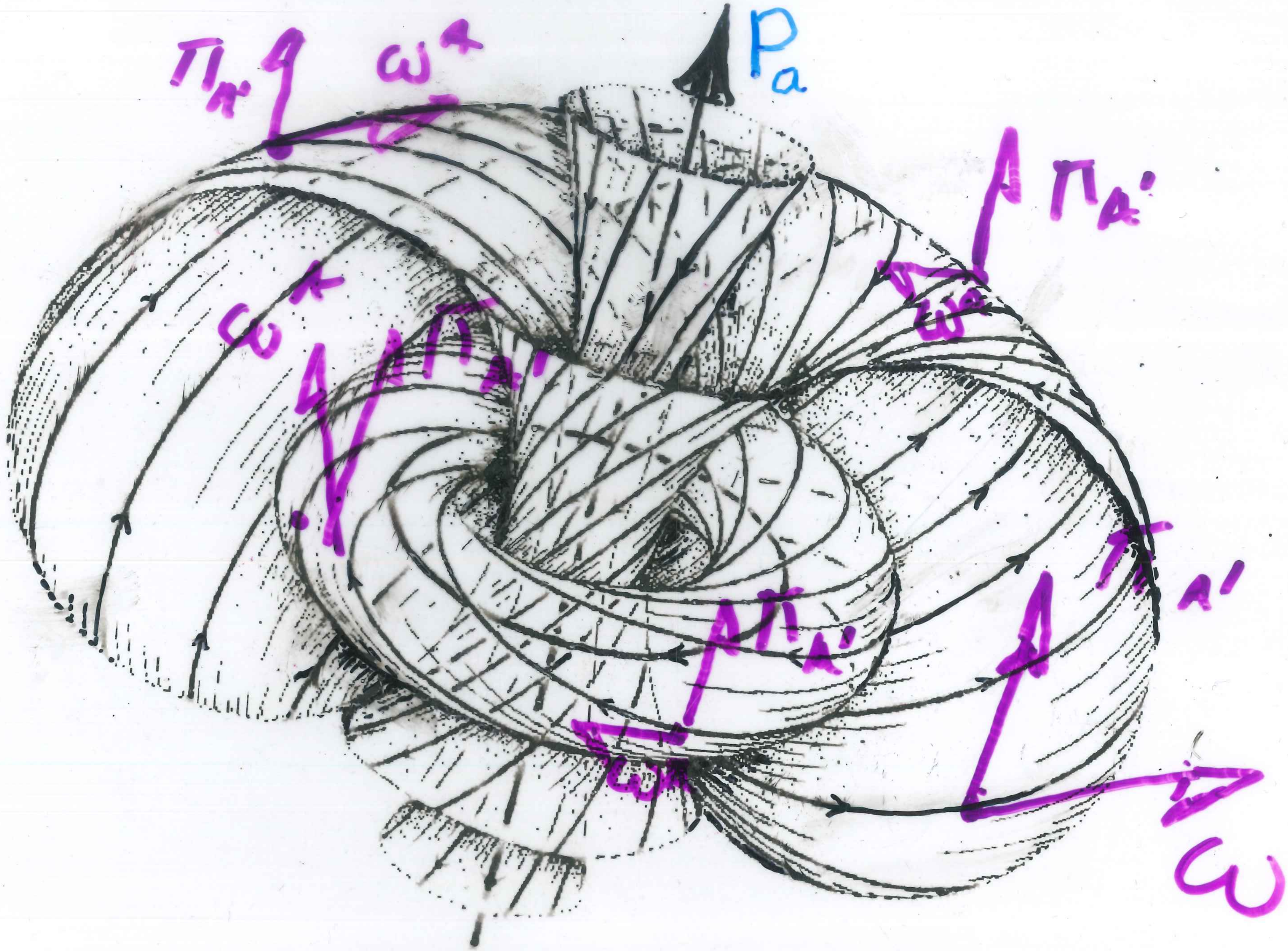
helicity

$$p_a = \pi_{A'} \bar{\pi}_A, \quad M^{ab} = i\omega^{(A} \bar{\pi}^{B)} \varepsilon^{A'B'} - i \varepsilon^{AB} \bar{\omega}^{(A'} \pi^{B')}, \quad 2s = Z^\alpha \bar{Z}_\alpha$$

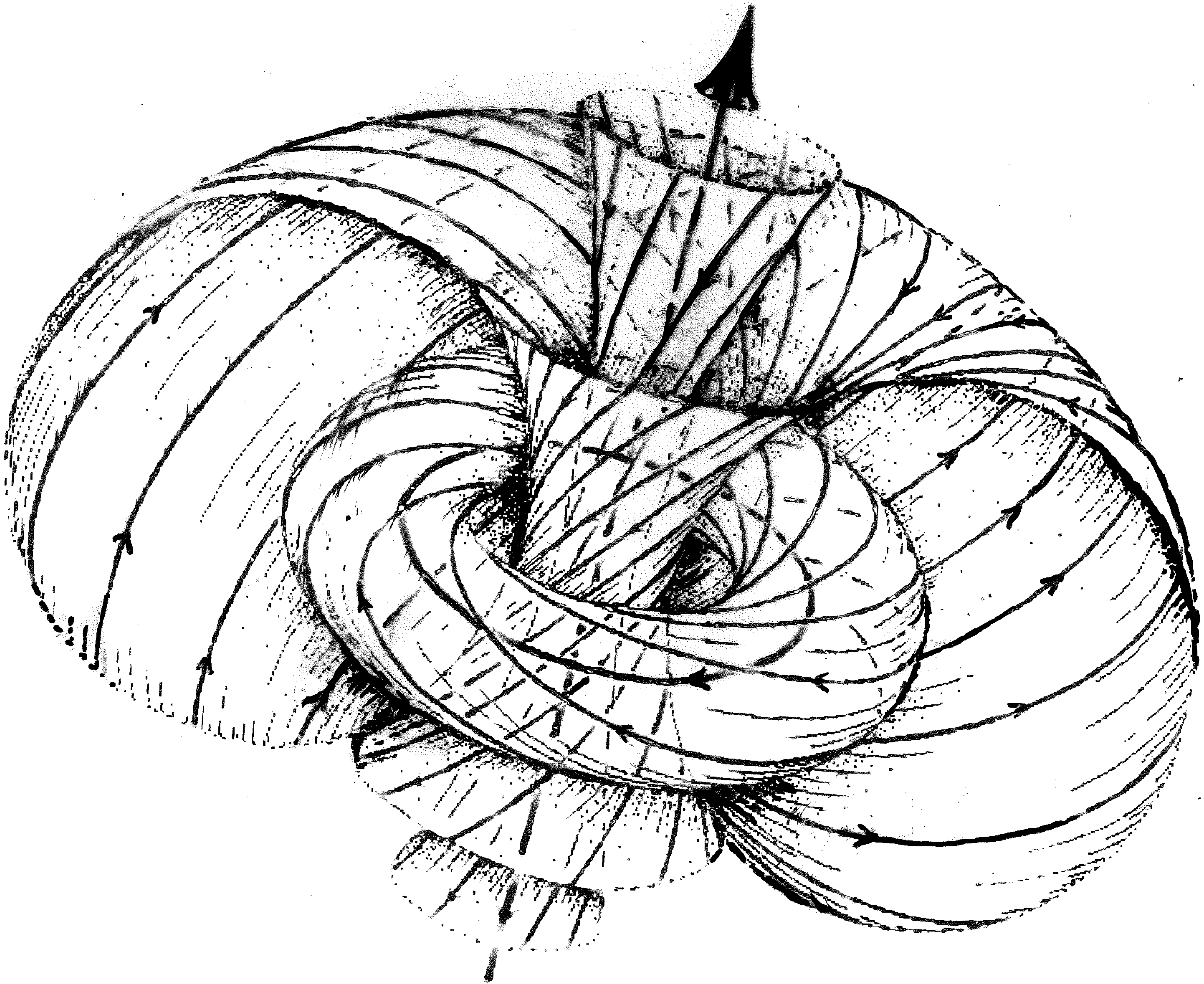
Note: (...) denotes symmetrization

P_a

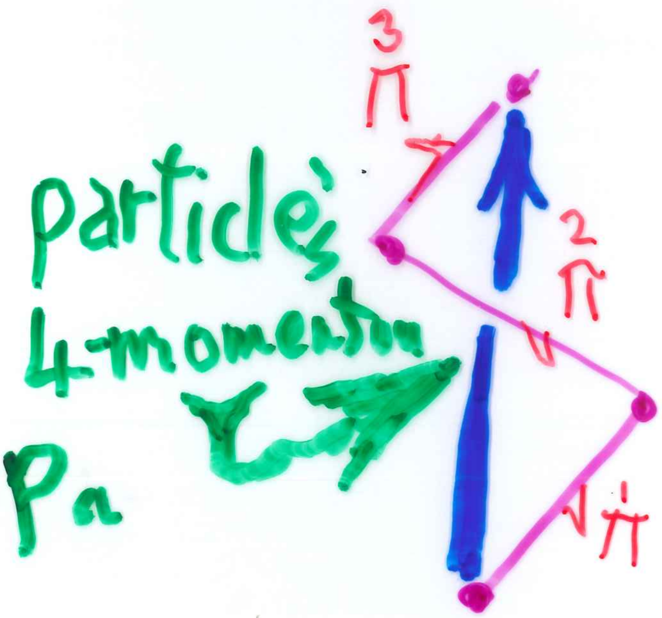




P_a
$\pi_{A'}$
ω^A
$\pi_{A'}$
ω^A
ω^A
$\pi_{A'}$
$\pi_{A'}$
$\pi_{A'}$
ω^A
ω^A



Massive particles



Use several twistors

Add their [momentum-angular mom] expressions, to get one for massive particle

Freedom in doing this gives twistor internal symmetry group

Suggests:

2 twistors: leptons

3 twistors: baryons

Massless Particles of Arbitrary Spin

Wave equations:

$s = 0$: $\quad \Box \phi = 0$

$s > 0$: $\quad \phi_{\underbrace{A'B'\ldots L'}_{2s}} = \phi_{(A'B'\ldots L')}$

$$\nabla^{AA'} \phi_{A'B'\ldots L'} = 0$$

$s < 0$: $\quad \phi_{\underbrace{AB\ldots L}_{2s}} = \phi_{(AB\ldots L)}$

$$\nabla^{AA'} \phi_{AB\ldots L} = 0$$

	2×helicity	Twistor hom.	Dual Twistor hom.
graviton	2	−6	2
photon	1	−4	0
massles anti-neutrino	1/2	−3	−1
Scalar	0	−2	−2
massles neutr.	−1/2	−1	−3
photon	−1	0	−4
graviton	−2	2	−6

Incidence: $\omega^{A} = i\, r^{AA'}\, \pi_{A'}$

Twistor: $Z^{\alpha} = (\omega^{A}, \pi_{A'})$

PN: $Z^{\alpha}\bar{Z}_{\alpha} = 0$, where $\bar{Z}_{\alpha} = (\bar{\pi}_{A}, \bar{\omega}^{A'})$

Linearized gravity (can be complex)

$$\nabla^{AA'}\psi_{ABCD} = 0, \qquad \nabla^{AA'}\tilde{\psi}_{A'B'C'D'} = 0.$$

Left-handed anti-self-dual

Right-handed self-dual

$$\psi_{ABCD}(r) = \oint_{\omega = ir\pi} \frac{\partial}{\partial\omega^{A}}\frac{\partial}{\partial\omega^{B}}\frac{\partial}{\partial\omega^{C}}\frac{\partial}{\partial\omega^{D}} f(Z)\, \pi\, d\pi$$

hom. deg +2

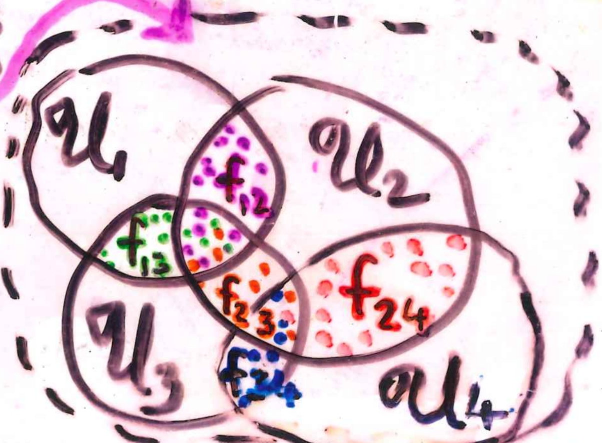
$$\tilde{\psi}_{A'B'C'D'}(r) = \oint_{\omega = ir\pi} \pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}\, \tilde{f}(Z)\, \pi\, d\pi$$

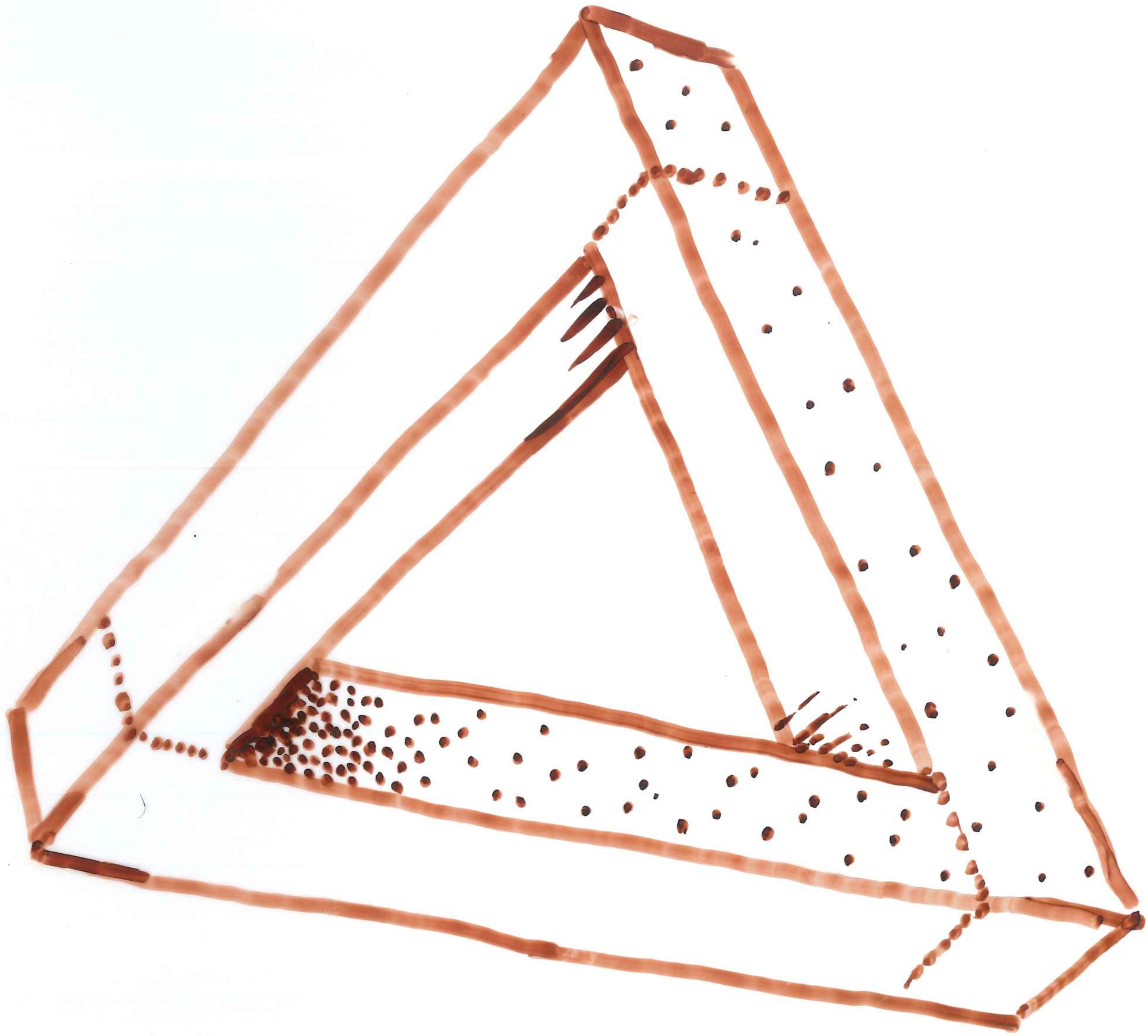
hom. deg. −6

covering $\{\mathcal{U}_i\}$

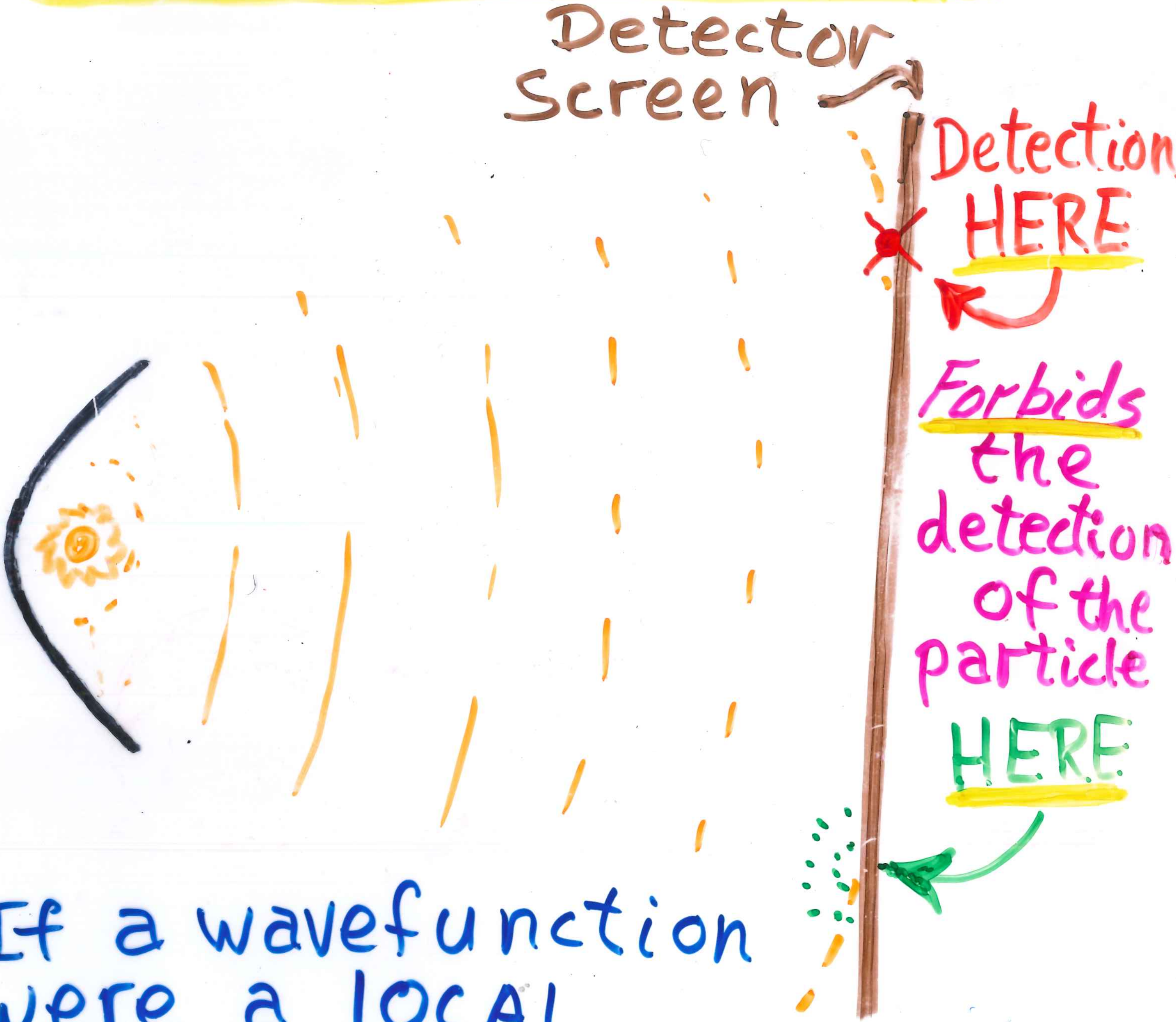
Really, f and $\tilde{f}$ are representatives of

COHOMOLOGY





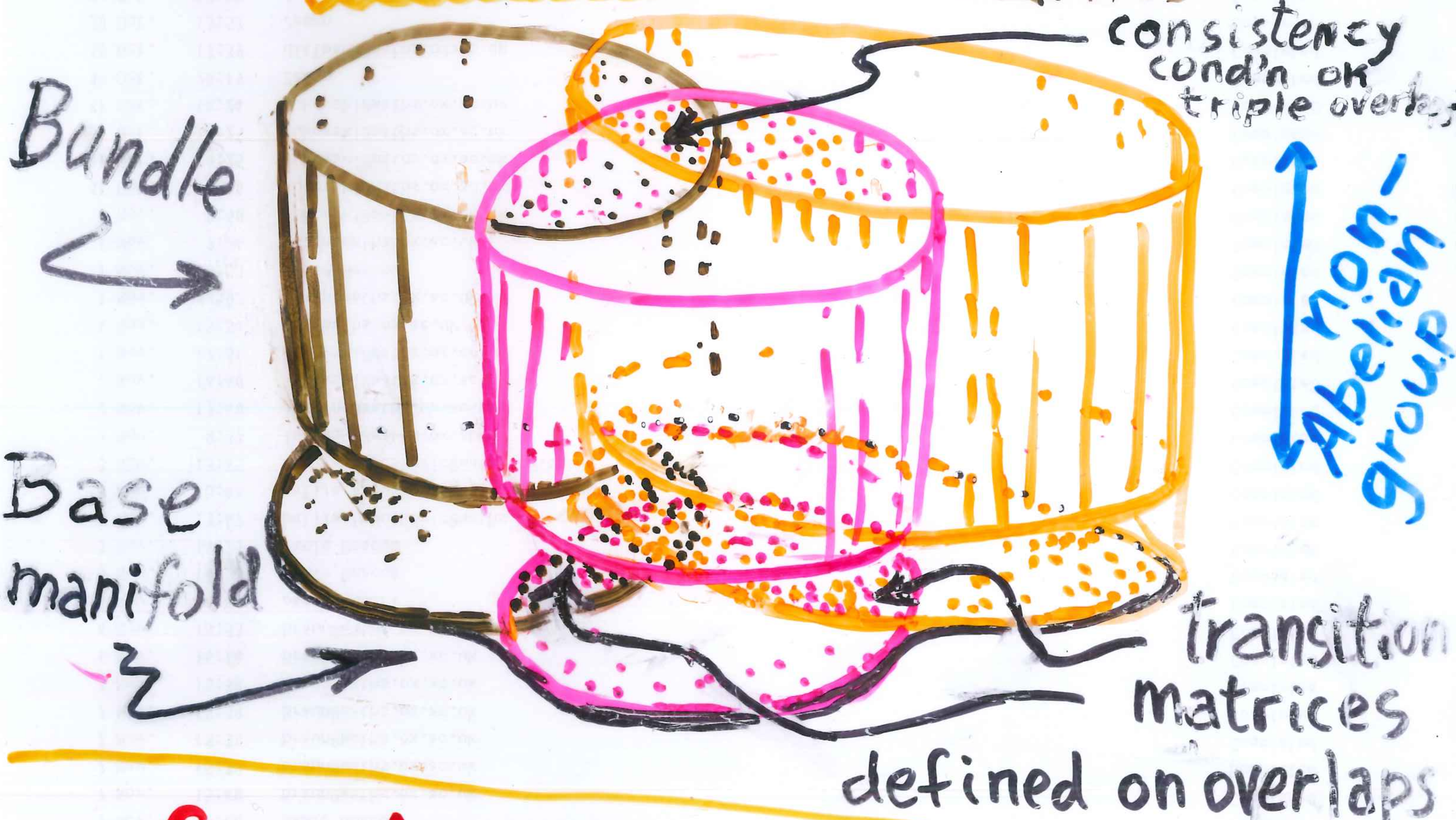
Non-Locality in the Wavefunction of a Single Particle



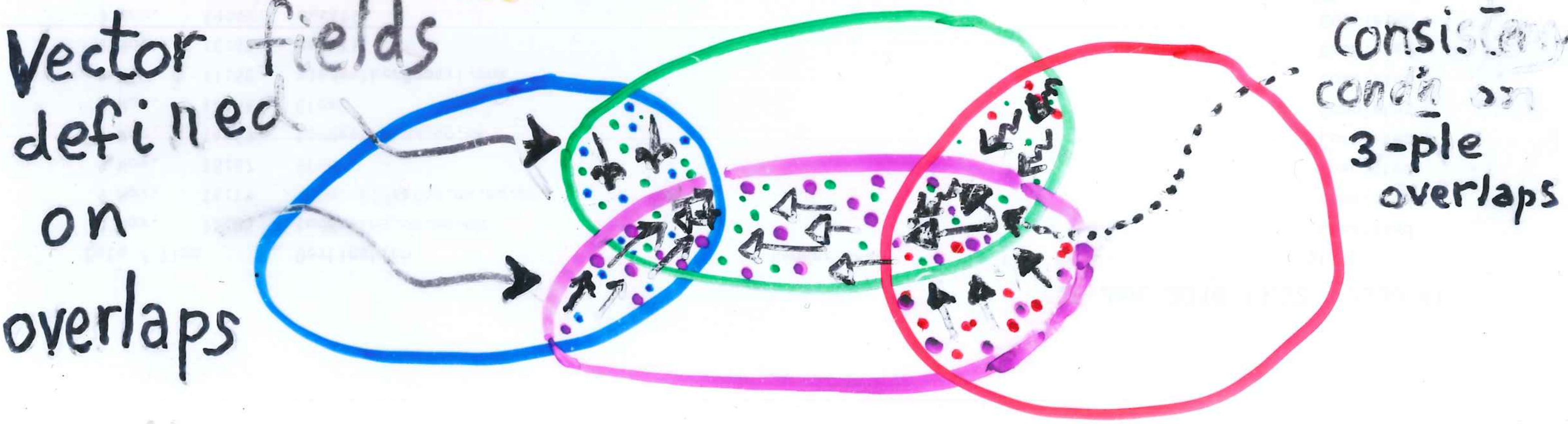
If a wavefunction were a LOCAL disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication

"Non-linearization" of 1^{st} (sheaf) Cohomology

• Construction of a bundle (Ward constr'n) (1977)



• Construction of a curved manifold (R.P. 1976)

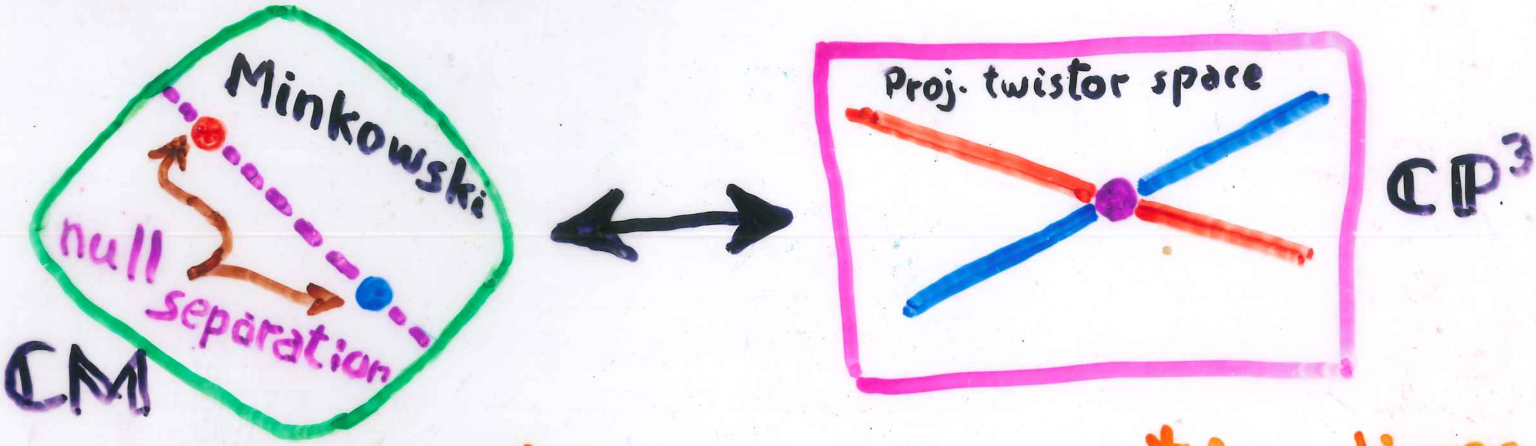


General Relativity

Numerous special applications
(e.g. Woodhouse-Mason: stationary axi-symm.)

As part of general programme:
"non-linear graviton construction" [R.P. '76]

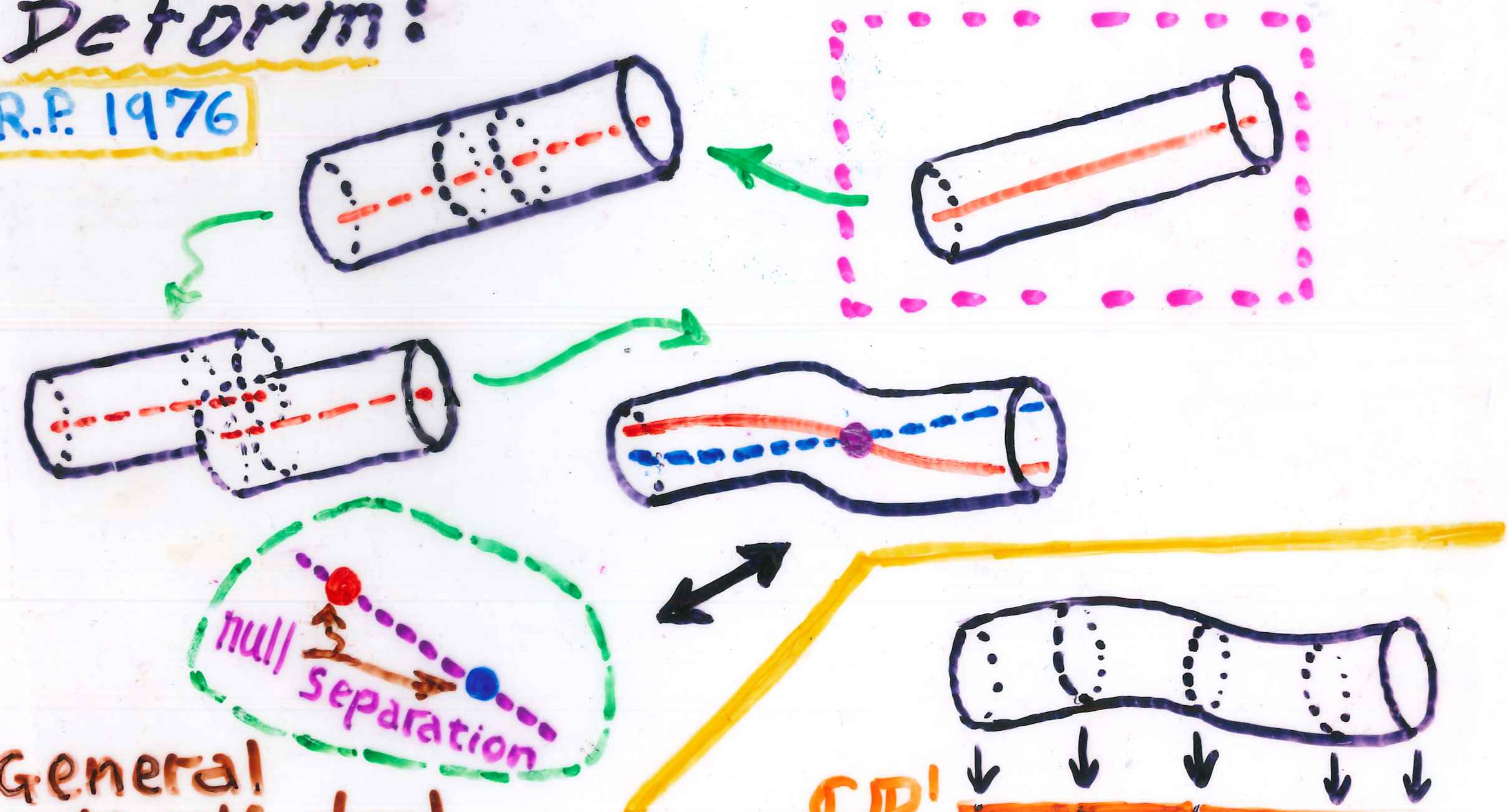
N.B. for flat space:



null separation ↔ meeting lines

Deform:

R.P. 1976



General anti-self-dual complex "space-time"

π-space

General anti-self-dual Ricci-flat complex "space-time"

Infinity twistors

Provide the space-time metric for Minkowski space $\mathbb{M}$ or (Anti) de Sitter $\mathbb{D}$

$$I_{\alpha\beta} = \begin{pmatrix} \frac{\Lambda}{6}\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}, \quad I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \frac{\Lambda}{6}\varepsilon_{A'B'} \end{pmatrix}$$

Λ = cosmological constant > 0

$I_{\alpha\beta} = \overline{I^{\alpha\beta}}$, $I_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} I^{\gamma\delta}$ Skew

$I^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} I_{\gamma\delta}$, $I_{\alpha\beta} I^{\beta\gamma} = -\frac{\Lambda}{6}\delta^{\gamma}_{\alpha}$

inverse if $\Lambda \neq 0$

Complex symplectic structure

$$\Theta = I_{\alpha\beta}\, dZ^{\alpha} \wedge dZ^{\beta} + I^{\alpha\beta}\, dW_{\alpha} \wedge dW_{\beta}$$

Sympl. potential: $I_{\alpha\beta} Z^{\alpha} dZ^{\beta} + I^{\alpha\beta} W_{\alpha} dW_{\beta}$

degenerate if $\Lambda = 0$

Googly Problem

(Note: a "googly" is a cricket ball bowled with an action that would appear to give it a left-handed spin, whereas the ball actually spins right handed.)

Geometrical coding of −6 (& −4) homogeneity

The "wavefunction" viewpoint, with regard to twistor functions demands that we adopt the chiral procedure of either a twistor ($Z^{\cdot}$) or a dual twistor ($W_{\cdot}$) description, rather than, say, an ambitwistor-type (($Z^{\cdot}, W_{\cdot}$) with $Z^{\alpha}W_{\alpha}=0$) procedure — which is more analogous to a classical-physics description.

Moreover, we cannot get away with using a different twistor space for each helicity, since we need to describe, say, plane-polarized photons, which are superpositions of the two

Quantum Twistor Theory

Z^α and $\bar{Z}_\alpha$ become non-commuting

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$

$$\bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar\, \delta^\alpha_\beta$$

So Z^α and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugates).

Choose $\hbar = 1$, for convenience. We find

$P_a = \pi_A \bar{\pi}_{A'}$, $\quad M^{ab} = i\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - i\varepsilon^{AB}\bar{\omega}^{(A'}\pi^{B')}$,

undisturbed by factor ordering, but

$$S = \frac{1}{4}\left(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha\right)$$

The standard commutators for P_a and M^{ab} follow

$P_a P_b - P_b P_a = 0$, $\quad P_a M^{bc} - M^{bc} P_a = i\,(g_a{}^b p^c - g_a{}^c p^b)$,

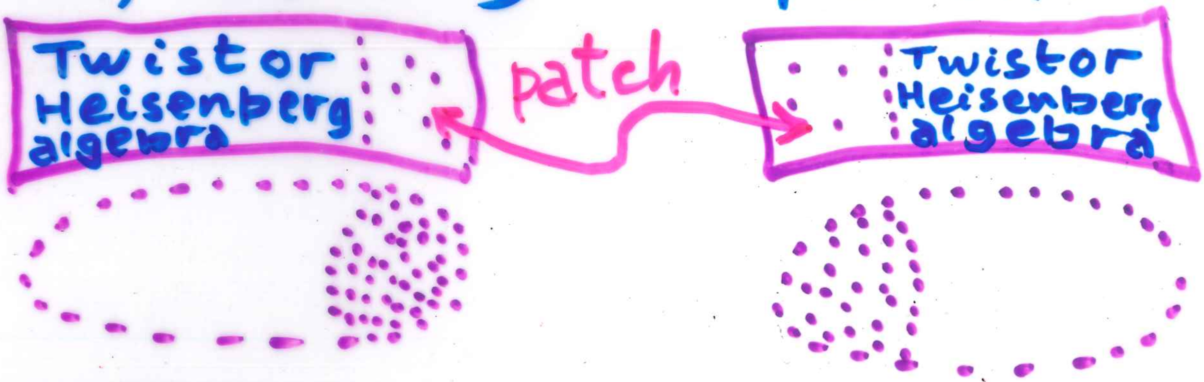
$M^{ab}M^{cd} - M^{cd}M^{ab} = i(g^{bc}M^{ad} - g^{bd}M^{ac} + g^{ad}M^{bc} - g^{ac}M^{bd})$.

Lie algebra generators for the Poincaré group.

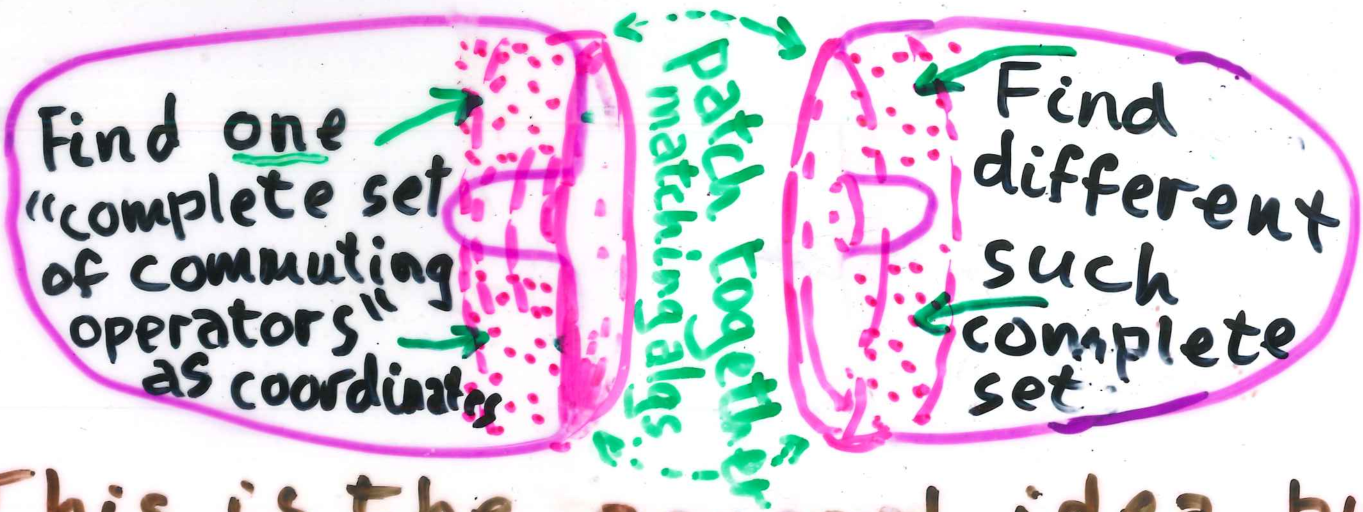
In twistor theory, holomorphicity plays a crucial role, and this is indeed retained through this twistor quantization procedure

$$\boxed{\bar{Z}_\alpha = -\partial/\partial Z^\alpha \,, \quad \partial/\partial \bar{Z}_\alpha = Z^\alpha} \quad \hbar = 1$$

Now, we try to patch:



The patching could proceed somewhat like this:



This is the general idea, but the patching could involve sublties

Twistor Geometric (pre-) Quantization

Symplectic form $\Sigma = i\, dZ^\alpha \wedge d\bar{Z}_\alpha = dp_a \wedge dx^a$
for a Lorentzian 4-manifold M

circle bundle

space $\mathbb{P}N$ of momentum-scaled light rays in M

In 3+1 dimensions with momentum

$$p_a = p_{AA'} = \bar{\pi}_A \pi_{A'}$$

the circle can be the phase freedom

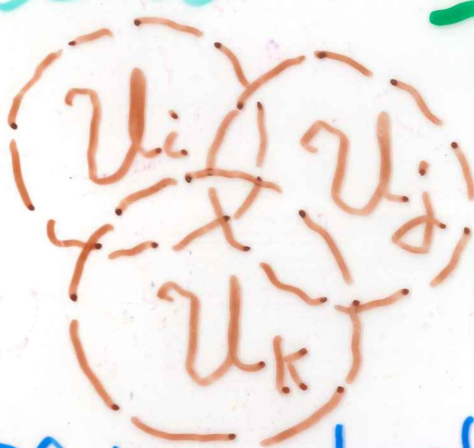
$$\pi_{A'} \rightarrow e^{i\theta} \pi_{A'}$$

We define a bundle connection

$$\nabla_a$$

whose curvature is Σ, but ∇_a is not canonically defined. The freedom in ∇_a is all important here

Palatial Twistor Theory



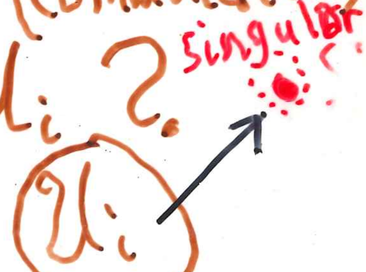
Try to patch algebras of functions on the "patches, not points".

If the algebras are commutative, we get nothing new, as points def'd by prime ideals

∴ need NON-commutative algebras MFA

∴ use the natural non-commutative twistor algebra $\mathbb{A}$, generated by Z^α and $\partial/\partial Z^\alpha$ ($\leftarrow$ in place of $\bar{Z}_\alpha$)

Technical issue: do we think of each $\mathbb{A}_i$ assigned to $\mathcal{U}_i$ as (appropriately smooth) subalgebra of $\mathbb{A}$ of linear operators on the (commutative) algebra of holomorphic functions on $\mathcal{U}_i$?



Problem: e.g. $e^{C^\alpha \partial/\partial Z^\alpha}$ dubious

whereas $e^{C_\alpha Z^\alpha}$ O.K.

& topology issues!

singular for $f(Z^\alpha)$ $Z^\alpha \in \mathcal{U}_i$

Geometric (pre-) Quantization

Take circle bundle $\mathcal{N}$ over the symplectic space $\mathbb{P}\mathcal{N}$, where $\mathcal{N}$ is space of spinor-scaled rays, where we take

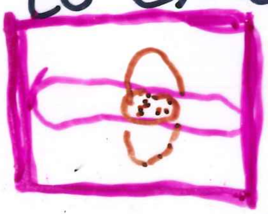
$p_a = \pi_{A'}\bar{\pi}_A$ (circle $\pi_{A'} \rightsquigarrow e^{i\theta}\pi_{A'}$)

Geometric (pre) quantization: look for bundle connections whose curvature is Σ.

Palatial twistor theory depends on this notion, but we need to envisage that this connection extends to non-null twistors. We can extend the 7-space $\mathcal{N}$, and there is no local obstruction to extending $\mathcal{N}$ to a flat twistor space $\mathbb{T}$ with the same 2-form Σ:

$$\Sigma = dp_a dx^a = i\, dZ^\alpha \wedge d\bar{Z}_\alpha$$

which can be done in many ways locally. Thus we can envisage a locally finite covering $\{\mathcal{U}_i\}$ of $\mathcal{N}$, where each $\mathcal{U}_i$ is (very non-uniquely) considered to extend into $\mathbb{T}$. Moreover, if we assume that



$\mathcal{M}$ is analytic (with analytic conformal structure, then we can complexify so that $\bar{Z}_\alpha$ becomes an independent dual twistor W_α.

Steps in the Palatial construction

Null-geodesic 5-space $\mathbb{P}\mathcal{N}$ exists for the Lorentzian globally hyperbolic 4-space $\mathcal{M}$. The momentum-scaled 6-space $\mathbb{p}\mathcal{N}$ is an $\mathbb{R}^+$-bundle over $\mathbb{P}\mathcal{N}$, where the null covector p_a points along the ray & is parallel-propagated along it.

$\mathbb{p}\mathcal{N}$ is a symplectic 6-manifold, with symplectic 2-form $\Sigma = dp_a \wedge dx^a$. In flat-space twistor theory, we can write this $\Sigma = i\, dZ^\alpha \wedge d\bar{Z}_\alpha$, and in an appropriate sense, this expression holds also in the palatial theory.

We wish to apply the procedures of geometric (pre-) quantization to $\mathbb{p}\mathcal{N}$ and for this we need an S^1-bundle over $\mathbb{p}\mathcal{N}$. Strikingly, this bundle is already to hand, as the phase factor in the spinor relation $p_{AA'} = \bar{\pi}_A \pi_{A'}$ $\pi_{A'} \mapsto e^{i\theta} \pi_{A'}$. In the flat case, this gives the twistor 7-space $\mathbb{N}$. In the general case we call it $\mathcal{N}$.

Explicit Construction of Algebra Generators from Geometric Quantization

The different bundle connections are provided by finding a 1-form Φ where

$$\Sigma = i\, d\Phi$$

and the bundle connection is then given by

$$\left(\frac{\partial}{\partial Z^\alpha} + P_\alpha,\ \frac{\partial}{\partial W_\alpha} + Q^\alpha\right)$$

and the two parts satisfy canon. commutation, where

$$\Phi = P_\alpha dZ^\alpha + Q^\alpha dW_\alpha$$

with

$$P_\alpha = -\frac{\partial E}{\partial Z^\alpha},\quad Q^\alpha = \frac{\partial F}{\partial W_\alpha}$$

where

$$E + F = Z^\alpha W_\alpha$$

(E F hol. in Z^α, W_α)

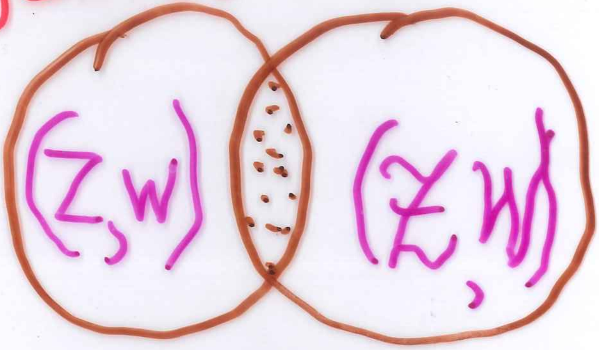
Particular case: $E=0$, $F = Z^\alpha W_\alpha$ gives $[Z^\alpha, Z^\beta] = 0$ $[\partial/\partial Z^\alpha, \partial/\partial Z^\beta] = 0$, $[Z^\alpha, \partial/\partial Z^\beta] = \delta^\alpha_\beta$

& connection takes form $\left(\frac{\partial}{\partial Z^\alpha}, \frac{\partial}{\partial W_\alpha} + Z^\alpha\right)$

Can take ket space hol. in Z^α & const. in W_α

Alternative, take $E = Z^\alpha W_\alpha$ $F = 0$ ← ket sp. hol. in W_α, const. in Z^α

Generating Functions: how to match algebras from one patch to another



We wish to preserve

$$\Sigma = dZ^\alpha \wedge dW_\alpha = d\mathcal{Z}^\alpha \wedge d\mathcal{W}_\alpha$$

on overlap

& Φ gives algebra generators

If we want to match the algebras, we need to know the relation between (Z^α, W_α) and $(\mathcal{Z}^\alpha, \mathcal{W}_\alpha)$

Take generating function $G(Z^\alpha, \mathcal{W}_\alpha)$ which is homogeneous of total degree 2. We have

$$\mathcal{Z}^\alpha = \frac{\partial G}{\partial \mathcal{W}_\alpha} \quad \text{and} \quad W_\alpha = \frac{\partial G}{\partial Z^\alpha}$$

Total homogeneity 2 means

$$\left(Z^\alpha \frac{\partial}{\partial Z^\alpha} + \mathcal{W}_\alpha \frac{\partial}{\partial \mathcal{W}_\alpha}\right) G = 2G$$

whence

$$Z^\alpha W_\alpha + \mathcal{Z}^\alpha \mathcal{W}_\alpha = 2G$$

and we have the required

$$\Sigma = dZ^\alpha \wedge dW_\alpha = d\mathcal{Z}^\alpha \wedge d\mathcal{W}_\alpha$$

Einstein's Λ-vacuum equations

we need (a) metric & (b) Λ-equations
This is achieved by preservation of

$$\Theta = dZ^\alpha \wedge dZ^\beta I_{\alpha\beta} + dW_\alpha \wedge dW_\beta I^{\alpha\beta},$$

where $I_{\alpha\beta} = \begin{pmatrix} \frac{\Lambda}{6}\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}$, $I^{\alpha\beta} = \begin{pmatrix} \varepsilon^{AB} & 0 \\ 0 & \frac{\Lambda}{6}\varepsilon_{A'B'} \end{pmatrix}$.

How do we preserve both 2-forms?
Take (total hom. deg. 2) function

$$\Gamma(\lambda Z^0, W_1, Z^3, \lambda W_2; W_0, \lambda Z^1, \lambda W_3, Z^2)$$

where $\lambda = i\sqrt{\frac{\Lambda}{6}}$

which is unchanged under interchange of 1st two entries & of 3rd and 4th entries (or of 5th & 6th and of 7th & 8th entries)

To get REAL LORENTZIAN $\mathcal{M}$, require $\Gamma \to \overline{\Gamma}$ under complete reversal of 1st four entries and of final four entries.

To get points of (real Lorentzian) $\mathbb{M}$, we find 4-dimensional completely commuting self-conjugate sub-algebra of $\mathcal{A}$

(like $\pi_{0'}, \pi_{1'}, \bar{\pi}_0, \bar{\pi}_1$ for origin in $\mathbb{M}$)

We need a version of Kodaira's theorem to see if this is sufficient characterization.

Why does preservation of Θ give us Einstein's Λ-equations?

Constancy of Θ under local twistor transport is equivalent to Λ-vacuum eqns.

So far, this is only classical, not quantum gravity, despite quantum commutators

Maybe we need:

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha + W_\rho W_\sigma \epsilon^{\alpha\beta\rho\sigma} = \zeta I^{\alpha\beta}$$

for ζ related to Planck length.