Twistor Theory

- Space
- Time
- Light ray
- Point (event)
- Space-time
- Complex
- Non-local
- Math. sophisticated

Riemann Sphere
Twistor space
Complex space

Lorentz group
Celestial spheres are Riemann spheres
Stereographic projection

Riemann sphere

Spin $\frac{1}{2}$ particle (e.g. electron, proton, neutron, quark)

$w + z = \gamma$

$\omega = \frac{z}{w}$

Riemann sphere
Positive/Negative Frequency Splitting

Riemann sphere $\mathbb{CP}^1$

Splitting of $H^0$ (ordinary functions) by holomorphic extension

$\leftarrow$ real axis

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$

Splitting of $H^1$'s, by holomorphic extension

$\leftarrow \mathbb{PN}$ (null twistors)

This realized (finally!) one of the very early motivations behind twistor theory
\[
\left( \frac{Z^0}{Z} \right) = i \sqrt{2} \left( \frac{r^0 + i r^3}{r^1 - i r^2} \right) \left( \frac{r^1 + i r^2}{r^0 - i r^3} \right) \left( \frac{Z^2}{Z^3} \right)
\]

\[
\omega^A = i \pi^A A'
\]

Incidence:
\[
\omega = i p \cdot \pi
\]

Shift of origin
\[
\omega^A = \omega^A - i q A' \pi^A A'
\]
\[
\pi_{A'} = \pi_A
\]

Null twistor
\[
Z^\alpha Z_\alpha = 0
\]

Equation of PN

Momentum-angular mom. for massless ptcle.
\[
P_a = 4\text{-momentum} \quad p_a p^a = 0 \quad p_a > 0
\]

M\text{\textsuperscript{ab}} = 6\text{-angular momentum}

Pauli-Lubanski spin vector
\[
S_a = \frac{i}{2} \epsilon_{abcd} P_b M_{cd} = S P a
\]

\[
P_a = \pi^A A' \bar{\pi}_A
\]

\[
M^{ab} = i \omega^{(A - B)} e^{A'B'} - i e^{AB'} \omega^{(A' A'' B'')}
\]

Note: (...) denotes symmetrization
Massive particles

Particle 4-momentum: $P_a$

Use several twistors

Add their momentum expressions to get one for massive particle freedom. In doing this gives twistor internal symmetry group

Suggests:
- 2 twistors: leptons
- 3 twistors: baryons
Massless Particles of Arbitrary Spin

Wave equations:

\[ S = 0 : \quad \Box \phi = 0 \]

\[ S > 0 : \quad \phi_{A'B'...L'} = \phi_{(A'B'...L')} \]

\[ \nabla^{AA'} \phi_{A'B'...L'} = 0 \]

\[ S < 0 : \quad \phi_{AB...L} = \phi_{(AB...L)} \]

\[ \nabla^{AA'} \phi_{AB...L} = 0 \]

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<tr>
<th>2\text{-helicity}</th>
<th>Twistor hom.</th>
<th>Dual twistor hom.</th>
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Incidence: \( \omega^A = i \gamma^{AA'} \pi_{AA'} \)

Twistor: \( Z^\alpha = (\omega^A, \pi_{AA'}) \)

PN: \( Z^\alpha \bar{Z}_\alpha = 0 \), where \( \bar{Z}_\alpha = (\pi_{AA'}, \omega^{AA'}) \)

Linearized gravity (can be complex)

Left-handed anti-self-dual

Right-handed self-dual

\[ \psi_{ABCD} = 0, \quad \hat{\psi}_{AB'C'D'} = 0 \]

\[ \psi_{ABCD}(r) = \oint \omega^A A^B A^C A^D f(Z) \pi d\pi \]

\[ \hat{\psi}_{AB'C'D'}(r) = \oint \pi_{A'A'} \pi_{B'B'} \pi_{C'C'} \pi_{D'D'} f(Z) \pi d\pi \]

Really, \( f \) and \( \hat{f} \) are representatives of Cohomology

Covering \( \{U_i\} \)
Non-Localilty in the Wavefunction of a Single Particle

If a wavefunction were a local disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication.
"Non-linearization of 1st (sheaf) cohomology"

- Construction of a bundle (Ward constr'n) (1977)
  - Consistency cond'n on triple overlaps
  - Abelian group
  - Transition matrices defined on overlaps

- Construction of a curved manifold (R.P. 1976)
  - Vector fields defined on overlaps
  - Consistency cond'n on 3-ply overlaps
General Relativity

Numerous special applications (e.g. Woodhouse-Mason: stationary axi-symm.)

As part of general programme: "non-linear graviton construction" [R.P. ’76]

N.B. for flat space:

![Diagram of Minkowski null separation and CP³ proj. twistor space](image)

null separation $\leftrightarrow$ meeting lines

Deform:

R.P. 1976

General anti-self-dual complex “space-time”
Infinity twistors

Provide the space-time metric for Minkowski space \([M]\) or \((\text{anti})\text{deSitter D}\)

\[
I_{ab} = \left( \frac{\Lambda}{6} \varepsilon_{ab} \quad 0 \right), \quad I = \left( \varepsilon_{ab} \quad 0 \right)
\]

\(\Lambda = \text{cosmological constant} \geq 0\)

\[
I_{ab} = \overline{I_{ab}}, \quad I_{ab} = \frac{1}{2} \varepsilon_{abst} I_{st}
\]

\[
I_{ab} = \frac{1}{2} \varepsilon_{abst} I_{st}, \quad I_{ab} I_{bc} = -\frac{\Lambda}{6} \delta_{ac}
\]

Complex symplectic structure

\[
\Theta = I_{ab} dz^a \wedge dz^b + I_{ab} dw^a \wedge dw^b
\]

Sympl. potential: \(I_{ab} Z^a dz^b + I_{ab} W^a dw^b\)

degenerate if \(\Lambda = 0\)
Googly Problem

(Note: a "googly" is a cricket ball bowled with an action that would appear to give it a left-handed spin, whereas the ball actually spins right handed.)

Geometrical coding of \(-6\) \((\& -4)\) homogeneity.

The "wavefunction" viewpoint, with regard to twistor functions demands that we adopt the chiral procedure of either a twistor \((Z; )\) or a dual twistor \((W; )\) description, rather than, say, an ambitwistor-type \((Z; W; )\) with \(Z^* W = 0\) procedure—which is more analogous to a classical-physics description.

Moreover, we cannot get away with using a different twistor space for each helicity, since we need to describe, say, plane-polarized photons, which are superpositions of the two...
Quantum Twistor Theory

$Z^a$ and $Z_{a}$ become non-commuting

$Z^a Z^b - Z^b Z^a = 0$

$Z_a Z_b - Z_b Z_a = 0$

$Z^a Z_b - Z_b Z^a = \hbar \delta^a_b$

So $Z^a$ and $Z_{a}$ are canonical conjugate variables (as well as complex conjugate). Choose $\hbar = 1$, for convenience. We find

$P_a = \pi_a \bar{\pi}_{a'}$, $M^{ab} = i\omega^{(A_{ab})} \varepsilon^{A'B'} - i\varepsilon^{A'B'} \bar{w}^{(A_{ab})}$,

undisturbed by factor ordering, but

$s = \frac{1}{4} (Z^a Z_a + Z_a Z^a)$

The standard commutators for $P_a$ and $M^{ab}$ follow

$[P_a, P_b - P_b P_a] = 0$, $P_a M^{bc} - M^{bc} P_a = i(g_{ab} p^c - g_{ac} p^b)$,

$M^{ab} M^{cd} - M^{cd} M^{ab} = i(g^{bc} M^{ad} - g^{bd} M^{ac} + g^{ad} M^{bc} - g^{ac} M^{bd})$.

Lie algebra generators for the Poincaré group.
In twistor theory, holomorphicity plays a crucial role, and this is indeed retained through this twistor quantization procedure:

\[ \bar{z}_x = -\partial / \partial z^x, \quad \bar{z} / \partial \bar{z}_x = z^x \] \( \hbar = 1 \)

Now, we try to patch:

Twistor Heisenberg algebra \( \rightarrow \) patch \( \rightarrow \) Twistor Heisenberg algebra

The patching could proceed somewhat like this:

Find one "complete set of commuting operators" as coordinates.

Patch together matching algs.

Find different such complete set.

This is the general idea, but the patching could involve subtleties.
**Twistor Geometric (pre-)Quantization**

Symplectic form $\Sigma=i\,dz^x_\alpha \wedge d\bar{z}^x_\alpha = dp_\alpha \wedge dx^\alpha$

for a Lorentzian 4-manifold $M$

The circle bundle

$\text{space of momentum-scaled light rays in } M$

In 3+1 dimensions with momentum $p_a = P_a A'$ = $\bar{\Pi}_a^A \Pi_{A'}$

the circle can be the phase freedom

$\Pi_{A'} \rightarrow e^{i\theta} \Pi_{A'}$ (؟)

We define a bundle connection $\nabla_\alpha$

whose curvature is $\Sigma$, but $\nabla_\alpha$ is not canonically defined. The freedom in $\nabla_\alpha$ is all important here
Palatial Twistor Theory

Try to patch algebra of functions on the "patches, not points."

If the algebras are commutative, we get nothing new, as points defined by prime ideals need NON-commutative algebras.

Use the natural non-commutative twistor algebra $\mathcal{A}$, generated by $\mathbb{Z}^*$ and $\mathfrak{A}_{\mathbb{Z}}^*$ (in place of $\mathbb{Z}$).

Technical issue: do we think of each $\mathcal{A}_i$ assigned to $U_i$ as (appropriately smooth) subalgebra of $\mathcal{A}$ of linear operators on the (commutative) algebra of holomorphic functions on $U_i$? Singular issues!

Problem: e.g. $e^{\alpha z^*}$ dubious & topology issues! whereas $e^{\alpha z^*}$ o.k.
**Geometric (pre-)Quantization**

Take circle bundle $N$ over the symplectic space $\mathbb{R}N$, where $N$ is space of spinor-scaled rays, where we take

$$p_a = \pi_a \pi_A$$

(circle $\pi_a \mapsto e^{i\theta} \pi_A$)

Geometric (pre) quantization: look for bundle connections whose curvature is $\Sigma$.

Talal twistor theory depends on this notion, but we need to envisage that this connection extends to non-null twistors. We can extend the 7-space $N$ and there is no local obstruction to extending $N$ to a flat twistor space $\mathbb{T}$ with the same 2-form $\Sigma$:

$$\Sigma = dP_a dx^a = i dZ^a d\overline{Z}^a$$

which can be done in many ways locally. Thus $N$, where each $\mathbb{U}$ is (very non-uniquely) considered to extend into $\mathbb{T}$. Moreover, if we assume that $M$ is analytic (with analytic conformal structure), then we can complexify so that $\Sigma$ becomes an independent dual twistor $\omega$.
Steps in the Palatini construction

Null-geodesic 5-space $\mathbf{PN}$ exists for the Lorentzian globally hyperbolic 4-space $\mathbf{M}$. The momentum-scaled 6-space $\mathbf{pN}$ is an $\mathfrak{T}^+\text{-bundle}$ over $\mathbf{PN}$, where the null covector $\mathbf{Pa}$ points along the ray & is parallel-propagated along it.

$\mathbf{pN}$ is a symplectic 6-manifold, with symplectic 2-form $\Sigma = dp_a \wedge dx^a$. In flat-space twistor theory, we can write this $\Sigma = i \partial Z^a \wedge \partial \bar{Z}_a$, and in an appropriate sense, this expression holds also in the palatial theory.

We wish to apply the procedures of geometric (pre-) quantization to and for this we need an $\mathbf{S}^1$-bundle over $\mathbf{pN}$. Strikingly, this bundle is already to hand, as the phase factor in the spinor relation $P_{aa'} = \Pi_A \Pi_{A'}$ $\Pi_{A'} \rightarrow e^{i\theta} \Pi_{A'}$. In the flat case, this gives the twistor 7-space $\mathbf{N}$. In the general case we call it $\mathbf{N}$. 
Explicit Construction of Algebra Generators from Geometric Quantization

The different bundle connections are provided by finding a 1-form $\Omega$ where

$$\Sigma = i \mathrm{d}\Omega$$

and the bundle connection is then given by

$$\Omega = P_a \mathrm{d}z^a + Q^x \mathrm{d}w_x$$

and the two parts satisfy canonical commutation, where

$$\Omega = P_a \mathrm{d}z^a + Q^x \mathrm{d}w_x$$

with

$$P_a = -\frac{\partial E}{\partial z^a}, \quad Q^x = \frac{\partial F}{\partial w_x}$$

where

$$E + F = z^a w_a$$

$$(E F \text{ hol. in } Z^a, W_a)$$

Particular case: $E = 0$, $F = z^a W_a$, gives

$$[z^a, z^b] = 0,$$ $[z^a, z^b] = 0,$ $[z^a, \bar{z}^b] = 0,$ $[\bar{z}^a, \bar{z}^b] = 0$,
and connection takes form

$$\left( \frac{\partial}{\partial z^a} \frac{\partial}{\partial w_x} + z^a \right)$$

Alternative, take $E = z^a W_a$, $F = 0$ < ket sp. hol. in $W$, const. in $z^a$.
Generating Functions: how to match algebras from one patch to another.

We wish to preserve
\[ \Sigma = dZ^\alpha dW_\alpha = d\tilde{Z}^\alpha d\tilde{W}_\alpha \]

If we want to match the algebras, we need to know the relation between \((Z, W)\) and \((\tilde{Z}, \tilde{W})\).

Take generating function \(G(Z, W_\alpha)\) which is homogeneous of total degree \(Z\). We have
\[ Z^\alpha = \frac{\partial G}{\partial W_\alpha} \quad \text{and} \quad W_\alpha = \frac{\partial G}{\partial Z^\alpha} \]

Total homogeneity 2 means
\[ (Z^\alpha \frac{\partial}{\partial Z^\alpha} + W_\alpha \frac{\partial}{\partial W_\alpha}) G = 2G \]

Whence
\[ Z^\alpha W_\alpha + Z^\alpha W_\alpha = 2G \]

And we have the required
\[ \Sigma = dZ^\alpha dW_\alpha = d\tilde{Z}^\alpha d\tilde{W}_\alpha \]
Einstein's $\Lambda$-vacuum equations

We need (a) metric & (b) $\Lambda$-equations. This is achieved by preservation of

\[ \Theta = dz^\alpha dz^\beta I_{\alpha\beta} + dw_\alpha dw_\beta I_{\alpha\beta} \]

where \( I_{\alpha\beta} = \left( \begin{array}{cc} 1 & \varepsilon_{AB} \\ 0 & \varepsilon_{AB} \end{array} \right) \), \( I_{\alpha\beta} = \left( \begin{array}{cc} \delta^{AB} & 0 \\ 0 & \delta^{AB} \end{array} \right) \).

How do we preserve both 2-forms? Take (total hom. deg. 2) function

\[ \Gamma (x, Z^0, W_1, Z^3, x, W_2, x, Z^1, x, W_3, x, Z^2) \]

where \( x = \sqrt{\frac{\Lambda}{6}} \)

which is unchanged under interchange of 1st two entries & of 3rd and 4th entries (or of 5th & 6th and of 7th & 8th entries) to get REAL LORENTZIAN $M$ require under complete reversal of 1st four entries and of final four entries.
To get points of (real Lorentzian) $M$, we find 4-dimensional completely commuting self-conjugate sub-algebra of $A$ (like $\Pi_0, \Pi_{\mathbb{I}}, \Pi_0, \Pi_{\mathbb{I}}$, for origin in $M$).

We need a version of Kodaira's theorem to see if this is sufficient characterization.

Why does preservation of $\mathbb{E}$ give us Einstein's $\Lambda$-equations?

Constancy of $\mathbb{E}$ under local twistor transport is equivalent to $\Lambda$-vacuum eqns.

So far, this is only classical, not quantum gravity, despite quantum commutators.

Maybe we need $Z^a Z^B Z^X + W^{a\rho} W^{b\sigma} E_{\rho\sigma} = \mathbb{E}$ for $\mathbb{E}$ related to Planck's length.