# Twistor geometry of rolling bodies

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# Motivation

- This research is motivated by the following fact established around the year 2000. (**R. Bryant, G. Bor, R. Montgomery, I. Zelenko, A. Agrachev, ...**)
- If one considers two balls of **different** radii $r$ and $R$, with $r < R$, rolling on each other, then such a kinematical system has **an obvious $\mathbf{SO}(3) \times \mathbf{SO}(3)$ global symmetry**. Even if we speacify that the sysytem is rolling **without slipping or twisting**, it still has this global symmetry.
- But if **the ratio of the radii** is $R : r = 3 : 1$, and **only** if this ratio is $3 : 1$, the dimmension of the local symmetry of such system jumps from $6$ to $14$, and what is even more surprising, the **local group of symmetries** of the system becomes isomorphic to the **split real form of the simple exceptional Lie group $G_2$**.

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# The simple exceptional Lie group $G_2$

- The simple exceptional Lie group $G_2$ is the **smallest** of the **five exceptional** simple Lie groups. They were predicted to exists by **Wilhelm Killing** in his classification of **complex** simple Lie groups obtained in 1887. It has **complex dimension** $14$.
- There are **only two real forms** of this group, the **compact** one , and the **noncompact** one, with the latter having **Killing form** of indefinite signature.
- It was predicted by Killing in **1887** that the **noncompact** form of $G_2$ can be realized as a **transformation group** on **five** dimensional manifolds.
- This prediction is true. The explicit realization was given independently by **Elie Cartan** and **Friedrich Engel** in **1893**. It is as follows:

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# Realization of $G_2$ by Cartan and Engel

- Consider an open set $\mathcal{U}$ of $\mathbb{R}^5$ with coordinates $(x, y, p, q, z)$ and a 2-plane $\mathcal{D}_{q^2}$ at each point of $\mathcal{U}$ spanned by two vector fields $X_1 = \partial_x + p\partial_y + q\partial_p + \frac{1}{2}q^2\partial_z$ and $X_2 = \partial_q$.
- This defines a rank 2 distribution $\mathcal{D}_{q^2} = \mathrm{Span}(X_1, X_2)$ on $\mathcal{U}$, called **Cartan-Engel distribution.**
- Search for local diffeomorphisms $\phi : \mathcal{U} \to \mathcal{U}$ such that $\phi_* \mathcal{D}_{q^2} = \mathcal{D}_{q^2}$. If such local diffeomorphisms exisit they form a **Lie group**, called the **group of local symmetries** of $\mathcal{D}_{q^2}$.
- Finding such diffeomorphisms, might be difficult, so search for their **infinitesimal versions**, i.e. vector fields $X$ on $\mathcal{U}$ such that $\mathcal{L}_X \mathcal{D}_{q^2} \subset \mathcal{D}_{q^2}$. If such vector fields exist they form a **Lie algebra**, actually, it is the **Lie algebra (of the group) of local symmetries** of $\mathcal{D}_{q^2}$.

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# Realization of $G_2$ by Cartan and Engel (continued)

- What is the Lie algebra of local symmetries of the Cartan-Engel distribution $\mathcal{D}_{q^2}$?
- Answer (**E. Cartan and F. Engel**):
  The Lie algebra $\mathfrak{g}$ of symmetries of $\mathcal{D}_{q^2}$ is a 14-dimensional **simple** real Lie algebra with not-definite Killing form.
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# Characterization of the Cartan-Engel distribution

- Given any distribution $\mathcal{D}$ on a manifold $M$ one considers a sequence $\mathcal{D}_{-1} = \mathcal{D}$, $\mathcal{D}_{-k-1} = [\mathcal{D}_{-1}, \mathcal{D}_{-k}]$, $k = 1, 2, 3, \dots$.
- At every point $x \in M$ the ranks $(r_{-1}, r_{-2}, \dots)$ of the respective spaces $\mathcal{D}_{-1}, \mathcal{D}_{-2}, \dots$ form a **growth vector** $(\mathcal{D}_{-1}, \mathcal{D}_{-2}, \dots)$ of the distribution $\mathcal{D}$. It is the simplest **local diffeomorphic invariant** of $\mathcal{D}$. Sometimes distributions have a **constant** growth vector.
- And sometimes there exists $k \in \mathbb{N}$ such that $r_{-k} = \mathrm{dim} M$.
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- At every point $x \in M$ the ranks $(r_{-1}, r_{-2}, \dots)$ of the respective spaces $\mathcal{D}_{-1}, \mathcal{D}_{-2}, \dots$ form a **growth vector** $(\mathcal{D}_{-1}, \mathcal{D}_{-2}, \dots)$ of the distribution $\mathcal{D}$. It is the simplest **local diffeomorphic invariant** of $\mathcal{D}$. Sometimes distributions have a **constant** growth vector.
- And sometimes there exists $k \in \mathbb{N}$ such that $r_{-k} = \mathrm{dim} M$.
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- The harmonic curvature of a $(2,3,5)$ distribution $\mathcal{D}$ on $M$ is called the **Cartan quartic**

$$C(\mathcal{D}) = \Phi_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = \Phi_0 + 4\zeta\Phi_1 + 6\zeta^2\Phi_2 + 4\zeta^3\Phi_3 + \zeta^4\Phi_4.$$

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# Goursat form

- It is result of **E. Gursat** that every $(2,3,5)$ distribution can be locally written in coordinates $(x,y,p,q,z)$ as $\mathcal{D}_F = \mathrm{Span}(X_1, X_2)$ with $X_1 = \partial_x + p\partial_y + q\partial_p + F\partial_z$ and $X_2 = \partial_q$, $F = F(x,y,p,q,z)$ such that $F_{qq} \neq 0$.
- If $F = f(q)$ then the Cartan quartic of the corresponding $(2,3,5)$ distribution is $C(\mathcal{D}_{f(q)}) = \zeta^4\Phi_4$ with

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# Summary of part I

- Among all $(2,3,5)$ distributions in dimension 5, there is a subclass, given up to local diffeomorphisms, whose Lie algebra of infinitesimal symmetries is isomorphic to the split real form of the simple exceptional Lie algebra $\mathfrak{g}_2$.
- Given a $(2,3,5)$ distribution $\mathcal{D}$ on a 5-manifold, one determines if it is locally diffeomorphically equivalent to the one with $\mathfrak{g}_2$ symmetry, by looking at the Cartan's quartic of $\mathcal{D}$.
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- What is the fundamental difference between $\mathbb{R}^4$ with a Riemannian metric $x_1^2+x_2^2+x_3^2+x_4^2$ and $\mathbb{R}^4$ with a Lorentzian metric $x_1^2+x_2^2+x_3^2-x_4^2$? Well ... in Lorentzian case we have **null vectors**, e.g. $n=(0,1,0,1)$.
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$$\big(\ a,\ \mathrm{Span}(n_1, n_2)\ \big) \mapsto \mathrm{Span}(a \cdot n_1, a \cdot n_2).$$

- Since the orthogonal group preserves nullity the resulting space $N_a^+ = \mathrm{Span}(a \cdot n_1, a \cdot n_2)$ is also totally null.
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# Real totally null planes (continued)

- Any totally null 2-plane $N = \mathrm{Span}(n_1, n_2)$ in $(\mathbb{R}^4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ defines a line of a bivector $l(N) = \mathbb{R} n_1 \wedge n_2$.
- It follows that the bivectors $l(N)$ are either selfdual: $*l(N) = l(N)$, or antiselfdual $*l(N) = -l(N)$.
- We say that a totally null plane $N$ is selfdual or antislefdual if its corresponding line $l(N)$ is selfdual or antislefdual, respectively.
- For example planes $N_\phi^+$ from the $\mathbf{SO}_0(2,2)$ orbit of $N_0^+$ are all selfdual.

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- The plane $N_0^- = \mathrm{Span}(n_1, n_3)$ with $n_1 = (1,0,1,0)$ and $n_3 = (0,1,0,-1)$ is antiselfdual.
- The entire $\mathbf{SO}_0(2,2)$ orbit of $N_0^-$, which is a cricle

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with $n_1(\phi) = (1, 0, \cos\phi, \sin\phi)$, $n_3(\phi) = (0, 1, \sin\phi, -\cos\phi)$, consists of antiselfdual planes.
- It follows that every totally null plane $N$ in $(\mathbb{R}^4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ belongs to either $\mathbb{S}^1_+$ or $\mathbb{S}^1_-$.
- The space $\mathcal{Z}(N)$ of all totally null planes in $\mathbb{R}^4$ equipped with the $(2,2)$ signature metric, is a disjoint union of $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$, $\mathcal{Z}(N) = \mathbb{S}^1_+ \cup \mathbb{S}^1_-$.

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# Circle twistor bundle

- Let $(M,g)$ be a 4-dimensional manifold $M$ equipped with a $(2,2)$ signature metric $g$. Assume that $M$ is orientable and oriented.
- Then, at every point $y \in M$ we have a circle $\mathbb{S}^1_+(y)$ of totally null selfdual planes $N^+_\phi(y)$ contained in the tangent space $\mathrm{T}_y M$.
- This defines a **circle bundle** $\mathbb{T}_+(M) = \cup_{y\in M}\mathbb{S}^1_+(y)$ with a projection: $\pi : N^+_\phi(y) \mapsto \pi(N^+_\phi(y)) = y$.
- The bundle $\mathbb{T}_+(M)$ is called a **circle twistor bundle** of a split-signature 4-manifold $(M,g)$.
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The bundle $\mathbb{T}(M)$ is very reach in geometric structures, which are induced on $\mathbb{T}(M)$ by the geometry of $(M, g)$. In particular:

- Vector fields tangent to the fibers of $\pi : \mathbb{T}(M) \to M$ form the **vertical space** $\mathcal{V}$ on $\mathbb{T}(M)$.
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- The horizontal rank 2 distribution $\mathcal{D}$ on $\mathbb{T}(M)$ as defined on the previous slide is called **twistor distribution** on $\mathbb{T}(M)$.
- Note that we found a natural **rank 2** distribution $\mathcal{D}$ on $\mathbb{T}(M)$, which is **five** dimensional.

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# Questions about the twistor distribution

- Immediately many questions arise:
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  - if $(2,3,5)$, then: when it is equivalent to the Cartan-Engel distribution $\mathcal{D}_{q^2}$?
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**Theorem**

*Twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ is integrable if and only if the split signature metric $g$ on $M$ has anti-selfdual Weyl tensor. Moreover, if the selfdual Weyl tensor of $g$ is nonvanishing in $\mathcal{U} \subset M$, then in $\pi^{-1}(\mathcal{U})$ there are open sets where the corresponding twistor distribution $\mathcal{D}$ is $(2,3,5)$.*

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*Twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ is integrable if and only if the split signature metric $g$ on $M$ has anti-selfdual Weyl tensor. Moreover, if the selfdual Weyl tensor of $g$ is nonvanishing in $\mathcal{U} \subset M$, then in $\pi^{-1}(\mathcal{U})$ there are open sets where the corresponding twistor distribution $\mathcal{D}$ is $(2,3,5)$.*

Let us assume that the selfdual part of the Weyl tensor of $g$ is not vanishig over an open set in $M$. Then, the corresponding twistor distribution $\mathcal{D}$ in $\mathbb{T}(M)$ is $(2,3,5)$, and the key question is: *which metrics $g$ have their twistor distributions locally equivalent to the Cartan-Engel distribution $\mathcal{D}_{q^2}$?* (the one with split $G_2$ symmetry).

This is a difficult question...But...

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# Results for a product of surfaces

## Theorem

*Let* $(\Sigma_1, g_1)$ *be a* **surface** *equipped* **with a Riemannian metric** $g_1$ *whose* **Gaussian curvature** *is* $\kappa$*, and such that it* **has a *Killing vector***, *and let* $(\Sigma_2, g_2)$ *be a* ***surface of constant Gaussian curvature*** $\lambda$*. Consider a 4-manifold* $M = \Sigma_1 \times \Sigma_2$ *with a product metric* $g = g_1 \oplus (-g_2)$*. Then in order for the twistor distribution* $\mathcal{D}$ *on* $\mathbb{T}(M)$ *to have local symmetry* $G_2$*, the curvatures must satisfy:*

$$(9\kappa - \lambda)(\kappa - 9\lambda)\lambda = 0, \qquad \kappa^2 + \lambda^2 \neq 0.$$

Obviously these equations can be satisfied only in **two cases**:

- the ratios of the curvatures are **1:9** or **9:1**, in which case both surfaces has constant curvatures,
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*If both surfaces $(\Sigma_1, g_1)$ and $(\Sigma_2, g_2)$ have constant Gaussian curvatures, respectively, $\kappa$, $\lambda$, then the Cartan quartic $C(\mathcal{D})$ of the twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ associated with $(M = \Sigma_1 \times \Sigma_2, g = g_1 \oplus (-g_2))$ is*

$$C(\mathcal{D}) = (9\kappa - \lambda)(\kappa - 9\lambda)h(\phi),$$

*where $h(\phi)$ is a nowhere vanishing function along the fibers of $\mathbb{T}(M)$*

Thus the cases when the ratio of constant curvatures is equal $1:9$ or $9:1$ correspond to twistor distributions with $G_2$ symmetry.

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## Corollary

*The twistor distributions $\mathcal{D}$ associated with the 4-manifold being a product of two spheres $\mathbb{S}^2$, whose radii are in the ratio $1:3$ or $3:1$ have $G_2$ symmetry.*

*The same is true for the product of two hyperboloids.*

I will comment on the remaining case $\lambda = 0$ and $(\Sigma_1, g_1)$ with Gaussian curvature $\kappa$ and Killing symmetry later.

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# Configuration space

We want to describe the space of possible positions for **two** (smooth) **rigid bodies** $B_1$ and $B_2$ that **roll on each other** in the 3-space $\mathbb{R}^3$.

- We idealize the surface of body $B_1$ by a surface $(\Sigma_1, g_1)$ with Riemannian metric $g_1$ and the surface of body $B_2$ by a surface $(\Sigma_2, g_2)$ with Riemannian metric $g_2$.
- To specify a position of the system, we chose a point $x$ on $\Sigma_1$ and a point $\hat{x}$ on $\Sigma_2$. These are the points in which the two surfaces kiss each other.
- To fully determine the possition of the system at a given time, we still need to fix the relative angle $\phi \in [0, 2\pi]$ between the tangent spaces $\mathrm{T}_x\Sigma_1$ and $\mathrm{T}_{\hat{x}}\Sigma_2$. This is equivalent to specify a rotation $A(\phi)$ which is an orthogonal transformation $A(\phi) : \mathrm{T}_x\Sigma_1 \to \mathrm{T}_{\hat{x}}\Sigma_2$ identifying the tangent spaces.

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- More formally the configuration space of the system is

$$\mathcal{T}(\Sigma_1, \Sigma_2) = \{ A(\phi) : \mathrm{T}_x\Sigma_1 \to \mathrm{T}_{\hat{x}}\Sigma_2 \},$$

clearly a **circle bundle** over the Cartesian product $M = \Sigma_1 \times \Sigma_2$, with fibers being circles $\mathbb{S}^1$ of orthogonal transformations $A(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

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$$\mathcal{T}(\Sigma_1, \Sigma_2) \;=\; \{\, A(\phi) : \mathrm{T}_x\Sigma_1 \to \mathrm{T}_{\hat{x}}\Sigma_2 \,\},$$

clearly a **circle bundle** over the Cartesian product $M = \Sigma_1 \times \Sigma_2$, with fibers being circles $\mathbb{S}^1$ of orthogonal transformations $A(\phi) = \begin{pmatrix} cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

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- There is a simple **bundle isomorphism** between the **configuration space** $\mathcal{T}(\Sigma_1, \Sigma_2)$ and **the twistor circle bundle** $\mathbb{T}(\Sigma_1 \times \Sigma_2)$.
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- A moment of reflexion yields

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$$\mathrm{graph}(A(\phi)) = (a, b, a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi)$$

$$= a(1, 0, \cos\phi, \sin\phi) + b(0, 1, -\sin\phi, \cos\phi))$$

$$= \mathrm{Span}\big(n_1(\phi), n_2(\phi)\big) = N^+_\phi \subset \mathbb{R}^4.$$

- This **identifies** bundles $\mathcal{T}(\Sigma_1, \Sigma_2)$ and $\mathbb{T}(\Sigma_1 \times \Sigma_2)$. This is to say that **the positions of the system of rolling bodies are totally null selfdual planes in the semi-Riemanian manifold** $M = \Sigma_1 \times \Sigma_2$, $g = g_1 \oplus (-g_2)$.

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- When the **surfaces** $\Sigma_1$ and $\Sigma_2$ **are rolling** on each other, they **draw a curve** $\gamma(t) = (x(t), \hat{x}(t), \phi(t))$ **in the configuration space** $\mathcal{T}(\Sigma_1, \Sigma_2)$, or via the described bundle isomorphism, a **curve** $\gamma(t) = (x(t), \hat{x}(t), N^+_\phi(t))$ **in the circle twistor bundle** $\mathbb{T}(\Sigma_1 \times \Sigma_2)$.
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- The tangency condition for the curve $\gamma$, means that the **velocity** $\dot{\gamma}$ of the system **is linearily restricted**; at each point $c = (x, \hat{x}, \phi)$ of the configuration space $\mathcal{T}(\Sigma_1, \Sigma_2)$ it is always at the 2-dimensional vector **subspace** $\mathcal{D}_c$ of the full 5-dimensional tangent space $\mathrm{T}_c\mathcal{T}(\Sigma_1, \Sigma_2)$.
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# Rolling without slipping or twisting

Suppose that we want to impose nonholonomic constraint of '**rolling without slipping or twisting**' on our system. For this we consider a curve $\gamma(t) = (x(t), \hat{x}(t), A(\phi(t)))$ in the configuration space $\mathcal{T}(\Sigma_1, \Sigma_2)$. It draws two curves: $x = x(t)$ on $\Sigma_1$, and $\hat{x} = \hat{x}(t)$ on $\Sigma_2$. These curves are just trajectories of the points of contacts.

- $\gamma(t)$ corresponds to the **movement without slipping** iff $A(\phi(t))\dot{x} = \dot{\hat{x}}$ – **two** linear conditions
- $\gamma(t)$ corresponds to the **movement without twisting** iff for every vector field $v(t)$ which is parallel along $x(t)$, the corresponding $A(\phi(t))$ transformed vector field $A(\phi(t))v(t)$ is parallel along $\hat{x}(t)$ – **one** linear condition.

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# Reinterpretation of the results about $M = \Sigma_1 \times \Sigma_2$

- **Circle twistor bundle** for the manifold $M = \Sigma_1 \times \Sigma_2$ with the metric $g = g_1 \oplus (-g_2)$ **is the configuration space of two rolling surfaces** $(\Sigma_1, g_1)$ and $(\Sigma_2, g_2)$.
- If the surfaces **roll** on each other '**without slipping or twisting**' their velocity space is restricted, in such a way that the **possible velocities** can only be **tangent to the twistor distribution.**
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- In paricular the system of **two spheres** of respective radii with ratio $1:3$ or $3:1$ which roll on each other without slipping or twisting has $G_2$ symmetry. But also the system of **two hyperboloids** of respective 'hyperbolic radii' with ratio $1:3$ or $3:1$ have this symmetry.
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- Well...Even the fact of $G_2$ symmetry for constant curvature surfaces with the curvature ratio $9:1$ is very surprising. The Cartan quartic of a $(2,3,5)$ distribution is a tensor with **five** components, so its vanishing for the twistor distribution $\mathcal{D}$ of two surfaces, requires a solution to **five nonlinear** PDEs. They are of $6^{th}$ order in terms of the derivatives of the metrics $g_1$ and $g_2$ of the rolling surfaces.This system of PDEs is **highly overdetrmined**, also because one part of the unknowns depends on $\Sigma_1$ only, whereas the other part depends on $\Sigma_2$ only. A lot of (very messy) equations appear, and it was hard to believe that they admit any solution.
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- Daniel An and me did not know all of this. We wrote the PDEs and were making assumptions on $g_1$ and $g_2$ to make them managable. In particular we simplified everything to the case when one surfaces has Killing symmetry, and the other has constant curvature.
- Let us thus examine the left case when $\lambda = 0$, and the other surface has a Killing symmetry.
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# Surfaces of revolution on the plane with $G_2$ symmetry

## Theorem

*Modulo homotheties all metrics corresponding to surfaces with a Killing vector, which when rolling on the* **plane** $\mathbb{R}^2$ *'without slipping or twisting', have the velocity distribution* $\mathcal{D}$ *with local symmetry* $G_2$ *are given by:*

$$\boxed{\begin{aligned} g_{1o} &= \rho^4 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1+} &= (\rho^2 + 1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1-} &= (\rho^2 - 1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \end{aligned}}$$

# Theorem (continued)

## Theorem

*or, collectively as:*

$$\boxed{g_1 = (\rho^2+\epsilon)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \quad \text{where} \quad \epsilon = 0, \pm 1.}$$

*Their curvature is given by*

$$\kappa = \frac{2}{(\rho^2+\epsilon)^3}.$$

# Surfaces of revolution on the plane with $G_2$ symmetry

**Theorem**

*Let $\mathcal{U}$ be a region of one of the Riemann surfaces $(\Sigma_1, g_1)$ of the previous Theorem, in which the curvature $\kappa$ is nonnegative. In the case $\epsilon = +1$, such a region can be isometrically embedded in flat $\mathbb{R}^3$ as a surface of revolution. The embedded surface, when written in the Cartesian coordinates $(X, Y, Z)$ in $\mathbb{R}^3$, is algebraic, with the embedding given by*

$$\boxed{(X^2 + Y^2 + 2)^3 - 9Z^2 = 0, \qquad \epsilon = +1.}$$

# Theorem (continued)

## Theorem

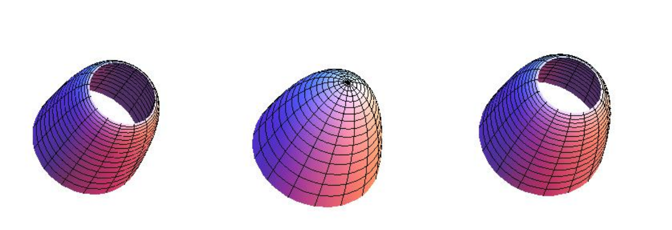
*In the case $\epsilon = -1$, one can find an isometric embedding in $\mathbb{R}^3$ of a portion of $\mathcal{U}$ given by $\varphi \in [0, 2\pi[$, $\rho \geq \sqrt{2}$. This embedding gives another surface of revolution which is also algebraic, and in the Cartesian coordinates $(X, Y, Z)$, given by*

$$\boxed{(X^2 + Y^2 - 2)^3 - 9Z^2 = 0, \qquad \epsilon = -1.}$$

*In the case $\epsilon = 0$, one can embed a portion of $\mathcal{U}$ with $\rho \geq 1$ in $\mathbb{R}^3$ as a surface of revolution*

$$\boxed{Z = f(\sqrt{X^2 + Y^2}), \quad \text{with} \quad f(t) = \int_{\rho=1}^{t} \sqrt{\rho^4 - 1}\, \mathrm{d}\rho.}$$

# How do they look?



Rysunek: The Mathematica print of the three surfaces of revolution, whose induced metric from $\mathbb{R}^3$ is given, from left to right, by respective metrics $g_{1-}$, $g_{1+}$ and $g_{1o}$. The middle figure embeds all $(\Sigma_1, g_{1+})$. In the left figure only the portion of $(\Sigma_1, g_{1-})$ with *positive* curvature is embedded, and in the right figure only points of $(\Sigma_1, g_{1o})$ with $\rho > 1$ are embedded. It is why the left and right figures have holes on the top. All three surface, when rolling on a plane 'without twisting or slipping' have velocity space $\mathcal{D}_v$ with symmetry $G_2$.

# An update from Robert Bryant

Dear Pawel,
I hope that this finds you well.
Igor Zelenko came to visit me this past week, and we talked a little bit about your $G_2$ rolling surface example in the context of doing computations for Cartan-type 2-plane fields.
It reminded me of the **left-over question of determining whether there are any other examples besides the constant curvature ones and your rotationally symmetric examples rolling over the plane**, so I took another look at the calculations and at the formula that I worked out for Cartan's C-tensor in this case.

# An update from Robert Bryant

It took a little thinking, but, based on this, **I now have a proof** (not too bad) **that, if a pair of Riemannian surfaces has the** $G_2$ **rolling distribution, then at least one of the two surfaces has to have constant Gauss curvature.**

# An update from Robert Bryant

I still don't know whether, if one fixes a constant Gauss curvature of one surface, the other surface has to have a rotational symmetry (which was your ansatz), but there is a clear line of attack for that, and, when I next have some time to look at this question, I'll see whether or not I can resolve it. **What is clear is that, for each fixed constant Gauss curvature of the one surface, there is at most a finite-dimensional space of isometry classes of germs of metrics that can roll over it with $G_2$ rolling distribution, and such a metric, if it exists, is completely determined by its 5-jet at one point.**

Yours,
Robert