Non-uniqueness of quantization,

Kähler geometry and generalized coherent state transforms

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On recent work in collaboration with
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Summary

$$(M,\omega) \rightsquigarrow \begin{cases} F & = & (F_1,\ldots,F_n) \rightsquigarrow (\omega, J_F, \gamma_F) \\ \mathcal{H}_F^{\mathsf{Q}} & = & \left\{ \Psi = \psi(F)\, e^{-k_F}, ||\Psi|| < \infty \right\} \subset \mathcal{H}^{\mathsf{prQ}} \\ F & \mapsto & \widehat{F}^{\mathsf{prQ}}|_{\mathcal{H}_F^{\mathsf{Q}}} = F \end{cases}$$

I. Ambiguity of quantization and preferred observables

I.1 Introduction

With $>$ 100 years of General Relativity and $>$ 90 years of Quantum Mechanics it is becoming increasingly embarassing the fact that there is not a fully consistent theory of Quantum Gravity.

The best candidates to succeed as e.g. Loop Quantum Gravity, String Theory, Causal Dynamical Triangulations, continue facing conceptual and technical problems.

One of the problems one is faced with in Loop Quantum Gravity and the one we will address today is that of nonuniqueness of quantization of a classical system.

In fact there is this whole ERC synergy grant - "Recursive and Exact New Quantum Theory" - lead by Andersen, Eynard, Kontsevich and Mariño, which has set as main goal to develop a new approach to quantum field theory in general.

The dream of the founders of quantum mechanics was to have quantization as a well defined process assigning a quantum system to every classical system and satisfying the correspondence principle

$$\text{Quantization Functor (?)} : (M, \omega) \mapsto Q_\hbar(M, \omega) \overset{\hbar \to 0}{\mapsto} (M, \omega)$$

It was soon realized that this can never be the case even for the simplest systems.

Particle in the line (1 dof)

Classical

$$
\begin{aligned}
(M,\omega) &= (\mathbb{R}^2, dq\wedge dp, H = \frac{1}{2}p^2 + V(q)),\\
f \rightsquigarrow X_f &= \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p} \qquad X_H = p\frac{\partial}{\partial q} - V'(q)\frac{\partial}{\partial p}
\end{aligned}
$$

Quantum

$$
\begin{array}{rcl}
 & Q_\hbar^{Sch}(\mathbb{R}^2, dp\wedge dq, H): & \\
\mathcal{H}^{\mathrm{Q}}_{(q)} \equiv \mathcal{H}^{\mathrm{Q}}_{Sch} & = & L^2(\mathbb{R}, dq) \\
q & \mapsto & Q_\hbar^{(q)}(q) = \widehat{q}^{Sch} = q \\
p & \mapsto & Q_\hbar^{(q)}(p) = \widehat{p}^{Sch} = i\hbar\frac{\partial}{\partial q} \\
f(q,p) & \mapsto & ?? \\
H = \frac{1}{2}p^2 + V(q) & \mapsto & Q_\hbar^{(q)}(H) = \widehat{H} = -\frac{\hbar^2}{2}\frac{\partial}{\partial q^2} + V(q)
\end{array}
$$

Groenewold (1946) – van Hove (1951) no go Thm:

It is impossible, even for systems with one degree of freedom, to quantize all observables exactly as Dirac hoped

$$\begin{aligned} Q_\hbar(f) &= \widehat{f} \\ [Q_\hbar(f), Q_\hbar(h)] &= i\hbar\, Q_\hbar(\{f,g\}) \end{aligned}$$

and satisfy natural additional requirements like irreducibility of the quantization.

In order to quantize one needs to add additional data to the classical system. *eg* choose a (sufficiently big but not too big ...) (Lie) subalgebra of the algebra of all observables

$$\mathcal{A} = \mathrm{Span}_{\mathbb{C}}\{1, q, p\}$$

Then we have to study the dependence of the quantum theory on the additional data.

I.2 Geometric Quantization

Geometric quantization is mathematically perhaps the best defined quantization

- Classical data: (M, ω), $\quad \frac{1}{2\pi\hbar}[\omega] \in H^2(M, \mathbb{Z})$

- Prequantum data: (L, ∇, h), $L \to M, F_\nabla = \frac{\omega}{\hbar}$

- Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M, L) = \overline{\left\{ s \in \Gamma^\infty(M, L) : ||s||^2 = \int_M h(s,s)\, \frac{\omega^n}{n!} < \infty \right\}}$$

- Pre-quantum observables: $\widehat{f} = Q_\hbar^{\mathrm{prQ}}(f) = \widehat{f}^{\mathrm{prQ}} = i\hbar\nabla_{X_f} + f$

This almost works! But the Hilbert space is too large, the representation is reducible.
We need a smaller Hilbert space: Prequantization $\Rightarrow$ Quantization

Additional Data in Geometric Quantization

Generalizing what is done in the Schrödinger representation, for systems with one degree of freedom, to fix a quantization one chooses (locally) a preferred observable – $F(q,p)^*$ – and then works with wave functions of the form

$$\mathcal{H}^{\mathsf{prQ}} \rightsquigarrow \mathcal{H}^{Q}_{(F)} = \left\{\Psi \in \mathcal{H}^{\mathsf{prQ}} : \nabla_{X_F}\Psi = 0, ||\Psi|| < \infty\right\} =$$
$$= \left\{\Psi(q,p) = \psi(F)\, e^{-k(q,p)},\ ||\Psi|| < \infty\right\} \subset \mathcal{H}^{\mathsf{prQ}}$$

on which the preferred observable F and functions of it $u(F)$ act diagonally

$$Q^{(F)}_{\hbar}(u(F)) = \widehat{u(F)}^{\mathsf{prQ}}|_{\mathcal{H}^Q_F} = u(F).$$

*for systems with n degrees of freedom one chooses (locally) n independent observables in involution $F = (F_1, \dots, F_n)$, $\{F_j, F_k\} = 0$. The distribution $\mathcal{P} =< X_{F_j}, j = 1, \dots n >$ is called polarization associated with this choice.

(Non–)Equivalence of different Quantizations

Are all these quantizations (for different choices of F) physically equivalent?

NO! Consider the following one parameter family of observables:

$$H_\lambda = \frac{p^2}{2} + \frac{q^2}{2} + \lambda \frac{q^4}{4}, \ \lambda \geq 0$$

and let $Sp(Q_\hbar^{(q)}(H_\lambda))$ denote the (discrete) spectrum of H_λ in the Schrödinger quantization, i.e. the spectrum of the operator

$$Q_\hbar^{(q)}(H_\lambda) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{q^2}{2} + \lambda \frac{q^4}{4}$$

acting on $\mathcal{H}^Q_{(q)} = L^2(\mathbb{R}, dq)$.

Now consider the 1–parameter family of quantizations with Hilbert spaces $\mathcal{H}^{\mathrm{Q}}_{H_\lambda}$ for which the role of preferred observable is played by H_λ. Then, one finds that

$$
\begin{aligned}
\mathcal{H}^{Q}_{(H_\lambda)} &= \left\{\Psi(q,p) : \nabla_{X_{H_\lambda}} \Psi = 0\right\} = \\
&= \left\{\Psi(q,p) = \psi(H_\lambda)\, e^{i\, G_\lambda(q,p)}\right\} = \\
&= \left\{\sum_{n=0}^{\infty} \psi_n\, \delta(H_\lambda - E_n^\lambda)\, e^{iG_\lambda(q,p)}\right\},
\end{aligned}
\tag{1}
$$

where E_n^λ are defined by the Bohr-Sommerfeld (BS) conditions

$$
\oint_{H_\lambda = E_n^\lambda} p dq = \hbar(n + \frac{1}{2}). \tag{2}
$$

Since $Q_\hbar^{(H_\lambda)}(H_\lambda) = H_\lambda$ we conclude from (1) that its spectrum in this quantization is given by (2) $Sp(Q_\hbar^{(H_\lambda)}(H_\lambda)) = \{E_n^\lambda, n \in \mathbb{N}_0\}$.

It is known that on one hand

$$Sp(Q_\hbar^{(q)}(H_0)) = Sp(Q_\hbar^{(H_0)}(H_0))$$

but on the other hand

$$Sp(Q_\hbar^{(q)}(H_\lambda)) \neq Sp(Q_\hbar^{(H_\lambda)}(H_\lambda)) = \text{BS semi-classical spectrum}$$

for all $\lambda > 0$ so that the two quantizations $Q_\hbar^{(q)}$ and $Q_\hbar^{(H_\lambda)}$ are physically inequivalent if $\lambda > 0$!

Wins $Q_\hbar^{(q)}$!

I.3 Ambiguity of quantization and reality conditions

LQG is facing a similar problem with the Ashtekar–Barbero connection as preferred observable

$$A_\beta = \Gamma + \beta\, K \Rightarrow \Psi_\beta(E, K) = \psi(A_\beta)\, e^{iG_\beta(E,K)}.$$

Are the quantizations based on the choice of connections with different (Barbero–Immirzi) parameters equivalent? No, because they lead e.g. to different spectra of the area operator.

Here it is less obvious which one is the "correct" one. Studies of the black hole entropy formula seemed to indicate the value

$$\beta = \ln(3)/\sqrt{8}\pi \;??$$

Other, recent studies (e.g. Pranzetti, Sahlmann, Phys.Lett 2015, Ben Achour, Livine, Phys. Rev. D 2017) however seem to point back to $\beta = \sqrt{-1}$. This corresponds to the Ashtekar connection

$$A_{\sqrt{-1}} = \Gamma + \sqrt{-1}K$$

It turns out that for some choices of complex observables quantization is in fact mathematically better defined then quantization based on real observables and this may help addressing some of the technical issues faced by LQG.

Complex observables and reality conditions: rescued by the power of complex analysis and complex geometry

Let us illustrate the general situation with a one degree of freedom system.

Consider the complex observable

$$z_f = q + if(p)\,, \qquad dz_f \wedge \overline{dz_f} = -2if'(p)\, dq \wedge dp\,.$$

It turns out that if $f'(p) > 0$ then several remarkable simplifying facts occur:

$$F_f = z_f = q + i f(p)$$

1. Complex Structure: There is a unique global complex structure J_f on $\mathbb{R}^2$ for which z_f is a global holomorphic coordinate:

$$J_f = i \frac{\partial}{\partial z_f} \otimes dz_f - i \frac{\partial}{\partial \bar{z}_f} \otimes d\bar{z}_f$$

2. Kähler Metric: The symplectic form together with the complex structure J_f define on $\mathbb{R}^2$ a Kähler metric

$$\begin{aligned} \gamma_f &= \frac{1}{f'(p)} \, dq^2 + f'(p) \, dp^2 \\ R(\gamma_f) &= -\left(\frac{1}{f'(p)}\right)'' . \end{aligned}$$

3. Quantum Hilbert space much better defined than in the case of quantizations based on real observables:

$$\mathcal{H}^{Q}_{(z_f)} = \left\{ \Psi(q,p) = \psi(z_f)\, e^{-k_f(p)},\ ||\Psi|| < \infty \right\} \subset L^2(\mathbb{R}^2)$$

where ψ is a J_f–holomorphic function and $k_f(p) = pf(p) - \int f(p)dp$ is a Kähler potential.

4. The inner product is not ambiguous and it fixes the reality conditions:

$$< \Psi_1, \Psi_2 >= \int_{\mathbb{R}^2} \overline{\psi_1(z_f)}\psi_2(z_f)\, e^{-2k_f(p)}\, dqdp\,.$$

II. Generalized CST transforms as liftings of geodesics from $\mathcal{T}_{Kah}$ to the quantum bundle

II.1 Applications of the (very rich) geometry of the space of polarizations ($\subset$ space of quantizations) $\mathcal{T}$

1. Kähler geometry: Donaldson–Tian theory of stability of Kähler manifolds. Finding new solutions to the geodesic equation on the space of Kähler metrics.

2. Quantization: CST for comparing different quantizations.

3. Quantum physics of open systems: Semiclassical approximation for complex Hamiltonians

4. Quantum Hall physics: Geometry dependence of Fractional Quantum Hall trial states – Laughlin states on curved surfaces.

5. Representation theory, algebraic geometry and quantization: Preferred basis of representations and their geometric interpretation (via BS fibers) and quantization with singular real and mixed polarizations

II.2 Quantum Bundle, geodesics and generalized CST

Let $\mathcal{T}$ be the space of polarizations ($\equiv$ choices of preferred observables $F \ \subset$ quantizations of M). In $\mathcal{T}$ we have the space of Kähler polarizations – $\mathcal{T}_{\text{Kah}}$ – and its boundary with real and mixed polarizations.

Geometric quantization gives us the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T} \supset \mathcal{T}_{\text{Kah}}$$

and the tools to study the dependence of quantization on the choice of F.

Integral transforms relating different quantizations

Step 1 Given two choices $F^{(1)}$ and $F^{(2)}$ we can hope to link them with a geodesic on $\mathcal{T}$, i.e. that there exists an Hamiltonian $H \in C^\omega(M)^*$ such that

$$F^{(2)} = e^{\tau\, X_H}|_{\tau=i}\, F^{(1)} = e^{t\, X_{iH}}|_{t=1}\, F^{(1)} \tag{3}$$

If both $F^{(1)}, F^{(2)} \in \mathcal{T}_{Kah}$ and the one-parameter family $e^{it\, X_H}\, F^{(1)}$, $t \in [0,1]$ is well defined, it defines a one parameter family of diffeomorphisms of M and a geodesics in $\mathcal{T}_{Kah}$ for the Mabuchi metric (see e.g. Donaldson, 1999 and M-Nunes, IMRN, 2015).

Step 2 Then, in principle, geometric quantization gives us a way of lifting the geodesics to the quantum bundle and thus construct an integral transform

$$U^{iH} \,:\, \mathcal{H}^{\mathrm{Q}}_{(F^{(1)})} \longrightarrow \mathcal{H}^{\mathrm{Q}}_{(F^{(2)})}$$

*more generally one considers complex Hamiltonians

Interpretation

Case 1 If the transform U^{iH} in Step 2 is unitary, as for $M = T^*K$, K a compact Lie group, $F^{(1)}$ corresponds to the Schrödinger (real) polarization and $F^{(2)}$ to the standard Kähler polarization (called adapted) for the bi-invariant metric on K and H is the norm square of the $K-$moment map, then one has established the equivalence of the two quantizations, $Q_\hbar^{(F_1)}$ and $Q_\hbar^{(F_2)}$.

Case 2 If not then we may still use the transform to study the difference of the two quantizations. In cases in which we have "preferred polarizations"(i.e. preferred quantizations) we may use the transforms in step 2 to "correct" other, nonpreferred, quantizations.

Some CST terminology

If in step 2 above the the preferred observables $F^{(1)}$ are real and $F^{(2)}$ correspond to a Kähler polarization then the integral transform is called a Coherent State Transform (CST) and H is called Thiemann complexifier. The name CST comes from the fact that they generalize the Segal–Bargmann CST for $M = \mathbb{R}^{2n}$

$$U^{i|p|^2/2} : L^2(\mathbb{R}^n, dx) \longrightarrow \mathcal{H}L^2(\mathbb{C}^n, e^{-|z|^2} dxdy)\,.$$

Thomas Thiemann introduced this formalism for quantum gravity in 1996. In general the transforms U^H, for complex H, are called generalized coherent state transforms.

II.3 Generalized Coherent State Transforms (gCST)

II.3.1 Hall transform

In 1994 Brian Hall constructed an unitary transform for Lie groups of compact type G

$$\begin{aligned} U \,:\, L^2(G, dx) &\longrightarrow \mathcal{H}L^2(G_{\mathbb{C}}, d\nu(g)) \\ U &= \mathcal{C} \circ e^{\frac{\Delta}{2}} \end{aligned} \tag{4}$$

where $G_{\mathbb{C}}$ is the unique complexification of G, $\mathcal{H}L^2$ means holomorphic L^2 functions and ν is the averaged heat kernel measure on $G_{\mathbb{C}}$.

II.3.2 Case $G=\mathbb{R}, M=T^*\mathbb{R}\cong\mathbb{R}^2$

Let us show how geometric quantization reveals the intimate relation of the two factors in the rhs of (4).

Then (4) reads

$$\begin{array}{rcl} U\,:\,L^2(\mathbb{R},dq) & \longrightarrow & \mathcal{H}L^2(\mathbb{C},e^{-p^2}dpdq) \\ U & = & \mathcal{C}\circ e^{\frac{\Delta}{2}} \\ \psi(q) & \mapsto & (e^{\frac{\Delta}{2}}\psi)(q)\mapsto(e^{\frac{\Delta}{2}}\psi)(q+\sqrt{-1}p)\,. \end{array}$$

Notice that, for $H=\frac{p^2}{2}$, $X_H=p\frac{\partial}{\partial q}$ and therefore

$$e^{\tau X_H}(q)|_{\tau=i}=(q+\tau p)|_{\tau=i}=q+ip=z$$

We see therefore that, for $H = \frac{p^2}{2}$,

$$\mathcal{C} = e^{iX_H}$$

and since $\widehat{H}^{\mathsf{prQ}} = iX_H - \frac{p^2}{2}$, we conclude that

$$e^{-i\tau \widehat{H}^{\mathsf{prQ}}}|_{\tau=i} = e^{\widehat{H}^{\mathsf{prQ}}} = \mathcal{C} \circ e^{-\frac{p^2}{2}}. \qquad \text{(changes the Hilbert space)}$$

On the other hand, since, $\widehat{p}^{\mathsf{Sch}} = -i\frac{\partial}{\partial q}$, we have also

$$e^{\frac{\Delta}{2}} = e^{-\widehat{H}^{\mathsf{Sch}}} = e^{-i\tau \widehat{H}^{\mathsf{Sch}}}|_{\tau=-i}, \qquad \text{(preserves the Hilbert space)}$$

We see therefore that the Hall CST transform in (4) is equivalent to the follwing transform lifting the complex canonical transformation, $e^{\tau X_H}|_{\tau=i} = e^{ip\frac{\partial}{\partial q}}$:

$$
\begin{aligned}
\mathcal{H}^{Q}_{\mathsf{Sch}} \equiv \mathcal{H}^{\mathsf{Q}}_{(q)} \;\; &\xrightarrow{U^{iH}} \;\; \mathcal{H}^{Q}_{(z)} = \mathcal{H}^{\mathsf{Q}}_{\mathsf{Fock}} \\
U^{iH} \;\; &= \;\; e^{-i\tau_1 \widehat{H}^{\mathsf{prQ}}}|_{\tau_1=i} \circ e^{-i\tau_2 \widehat{H}^{\mathsf{Sch}}}|_{\tau_2=-i} = \\
&= \;\; e^{+\widehat{H}^{\mathsf{prQ}}} \circ e^{-\widehat{H}^{\mathsf{Sch}}}
\end{aligned}
\tag{5}
$$

with the (extra bonus of the) averaged heat kernel measure being absorbed into the prequantization of the complexified canonical transformation.

II.3.3 Representation Theoretic meaning of the factors in the (abelian) CST

Notice that the prequantization of the observables q, p preserve both Hilbert spaces $\mathcal{H}^{Q}_{\mathsf{Sch}} = \mathcal{H}^{\mathsf{Q}}_{(q)}$ and $\mathcal{H}^{Q}_{\mathsf{Fock}} = \mathcal{H}^{\mathsf{Q}}_{(z)}$ so that there is a $*-$representation of the complexified Heisenberg algebra on both.

One can check that the first factor to act in (5) maps the self-adjoint $\widehat{q}^{\mathsf{Sch}}$ to the non self-adjoint $\widehat{q - ip}^{\mathsf{Sch}}$ and the second factor to act maps $\widehat{q}^{\mathsf{Sch}}$ to $\widehat{q + ip}^{\mathsf{Fock}}$ and therefore U^{iH} maps $\widehat{q}^{\mathsf{Sch}}$ to $\widehat{q}^{\mathsf{Fock}}$ (and $\widehat{p}^{\mathsf{Sch}}$ to $\widehat{p}^{\mathsf{Fock}}$).

Then U^{iH} intertwines $\widehat{q}^{\mathsf{Sch}}$ and $\widehat{p}^{\mathsf{Sch}}$ with $\widehat{q}^{\mathsf{Fock}}$ and $\widehat{p}^{\mathsf{Fock}}$, respectively, which makes its projective unitarity a consequence of Schur's lemma.

$t = is$ can change the type of a polarization

Using one–parameter "groups" of complex " canonical transformations" generated by complex Hamiltonians H we can change the type of the polarization (from real or mixed to Kähler and back).

A (finite) time (say $t = 1$) complex symplectomorphism generated by the (complex) function H transforming a real polarization to a Kähler polarization corresponds algebraically to the analytic continuation of functions from a Lagrangian submanifold $Y \subset M$ (e.g when $M = T^*Y$) to M with a given complex structure I. Geometrically its the inverse of the collapse of (M, I) to Y. This is literally so as the complex symplectomorphism generated by the same function H, at time $t = -1$, corresponds to the collapse of (M, I) to Y.

$t = i\infty$ **can change the topology of a polarization**

Let $M = T^*S^1 \ni (\theta, p)$ and let us show that imaginary time canonical transformations generated by $H = |\mu|^2/2 = |p|^2/2$ (with $X_H = p\frac{\partial}{\partial\theta}$) take us from $\mathcal{P}^{\mathsf{Sch}}$ to $\mathcal{P}^{\mathsf{mom}}$ in infinite imaginary time $t = i\infty$.

$$e^{it\,\mathcal{L}_{X_H}}\langle X_\theta\rangle = \langle X_{\theta+itp}\rangle = \langle \frac{1}{it}X_\theta + X_p\rangle \xrightarrow{\mathsf{t}\to\infty} \langle X_p = \frac{\partial}{\partial\theta}\rangle$$

It turns out that this limit can be extended to a very wide context.

A$_1$ For $M = T^*T^n$, $H = |p|^2/2$ and for all (infinite dimensional (!) space of) starting real, mixed or Kähler polarizations $\mathcal{P}$ for which the limit

$$\lim_{t\to\infty} e^{it\,\mathcal{L}_{X_H}}\,\mathcal{P}$$

exists, it is equal to the momentum polarization, $\mathcal{P}^{\mathrm{mom}}$.

A$_2$ Let U/K be a symmetric space of compact type. The limit above can be extended to an infinite dimensional family of U–invariant Kähler polarizations on $T^*(U/K)$ and an infinite dimensional space of μ–convex U–invariant initial velocities of the Kähler potential, $\dot{k}_0 = -2H$, e.g. $H = ||\mu||^2$.

The limit is a natural (mixed) polarization which we call Kirwin–Wu, $\mathcal{P}^{\mathrm{KW}}$, due to its discovery by Kirwin and Wu,

$$\mathcal{P}^{\mathrm{KW}} := \lim_{t\to\infty} \mathcal{P}^t = \lim_{t\to\infty} e^{it\,\mathcal{L}_{X_H}}\,(\mathcal{P}) \qquad \forall \mathcal{P} \in \mathcal{P}^{\mathrm{Kah}}_G$$

In particular we get [Kirwin–Wu (unpublished)] and [Baier–Hilgert-Kaya-M-Nunes (reproved and extended to an infinite dimensional space of polarizations and initial Kähler velocities on cotangent bundles of compact symmetric spaces)] that $\mathcal{P}^{\mathsf{KW}}$ is a strongly integrable mixed polarization generated by collective integrals called Guillemin–Sternberg action variables and complex valued functions on $\mathbb{S}_{\lambda+\rho} = \mu^{-1}(\mathcal{O}_{\lambda+\rho})$ that are pullbacks of meromorphic functions on the U–coadjoint orbits $\mathcal{O}_{\lambda+\rho}$. For $\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}^{\mathsf{KW}}}$ we get

$$\mathcal{H}^{\mathsf{Q}}_{\mathcal{P}^{\mathsf{KW}}} = \sum_{\lambda \in \Lambda_+} \delta_{\mu^{-1}(\mathcal{O}_{\lambda+\rho})} \, H^0(L_{\lambda+\rho})$$

where ρ denotes the half-sum of positive roots of U projected to the space generated by K–spherical weights.

Some of our recent papers on this subject:

- T. Baier, J. Hilgert, O. Kaya, J.Mourão and J.P. Nunes, *Compact symmetric spaces, partial Bohr–Sommerfeld leaves and infinite μ–convex geodesics in the space of Kähler metrics*, to appear soon.

- W. Kirwin, J.Mourão, J.P. Nunes and T. Thiemann, *Segal-Bargmann transforms from hyperbolic Hamiltonians*, arXiv:2019.

- W. Kirwin, J.Mourão and J.P. Nunes, *Complex symplectomorphisms and pseudo-Kahler islands in the quantization of toric manifolds*, Math Analen **364** (2016) 1–28.

- J.Mourão and J.P. Nunes, *On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics*, Int Math Research Notices (2015) 10624–10656

Congratulations Jurek !!!