Scattering on plane waves and the double copy

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Joint work with Adamo, Casali & Nekovar 2017-8, arxiv:1706.08925, 1708.09249, 1810.05115.
Further work to come with Tim Adamo & Atul Sharma.

Calculate amplitudes on plane-wave backgrounds.

Conventional motivation:

- Construct interacting perturbative QFT on asymptotically simple curved backgrounds.
- Plane waves are universal as Penrose limits.
- Computable: Plane waves satisfy Huygens & have separable Hamilton-Jacobi and linear fields.
- Test Gravity = Double copy of Yang-Mills.

We needed data to check 3 point curved background ambitwistor string YM & gravity formulae.

[Adamo, Casali, M & Nekovar 1708.09249]

[Aim: extend amplitude & ambitwistor-strings to curved space.]

Yang-Mills amplitudes & colour-kinematic duality

Scatter n particles, momentum k_μ, polarization ϵ_μ

$$A_\mu(x) = \epsilon_\mu \mathrm{e}^{ik\cdot x} t^a , \qquad k^2 = 0 , \quad k \cdot \epsilon = 0 , \quad t^a \in \text{Lie } G.$$

- Suppose that YM amplitude $\mathcal{A}(k_i, \epsilon_i, t_i)$, $i = 1, \dots, n$ arises from trivalent Feynman diagrams

$$\mathcal{A} = \sum_\Gamma \frac{N_\Gamma(k_i, \epsilon_i) C_\Gamma(t_i)}{D_\Gamma} , \qquad \Gamma \in \{ \text{ trivalent diagrams, n legs} \} .$$

- $N_\Gamma =$ *kinematic factors*: polynomials in k_i, linear in each ϵ_i.
- $D_\Gamma = \prod_{propagators\ e \in \Gamma} (\sum_{i \in e} k_i)^2 =$ *denominators*.
- $C_\Gamma(t_i) =$*colour factor* = contract structure contants at each vertex together along propagators and with t_i at ith leg.

Definition

The N_Γ are said to BCJ numerators if N_Γ satisfy identities when C_Γ does via Jacobi identities: $C_{\tilde\Gamma} = C_\Gamma + C_{\Gamma'} \Rightarrow N_{\tilde\Gamma} = N_\Gamma + N_{\Gamma'}$.

Possible at tree-level and up to 4-loops, but not canonical.

Gravity as double copy of Yang-Mills

Zvi Bern, J J Carrasco, H Johansson, 2008

Scatter *n* gravity plane waves

$$h_{\mu\nu} = \epsilon_{(\mu}\epsilon_{\nu)}\mathrm{e}^{ik\cdot x}$$

Given BCJ numerators N_Γ, the gravity tree-amplitude/loop integrand can be obtained as a *double copy* of YM amplitude

$$\mathcal{M}(k_i, \epsilon_i, \epsilon_i) = \sum_\Gamma \frac{N_\Gamma(k_i, \epsilon_i) N_\Gamma(k_i, \epsilon_i)}{D_\Gamma}$$

- KLT tree relation gravity amplitudes $=$ (YM)^2 from strings.
- Proved up to 4-loops.
- There are extensions to many theories.
- Genuine tool for constructing gravity amplitudes.
- No nonperturbative or space-time explanation.

The three point amplitude

At three points, there is just one trivalent diagram

$$\mathcal{A} = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot (k_1 - k_2) + \circlearrowleft) f_{abc} t_1^a t_2^b t_3^c = N_{\curlywedge}(\epsilon_i, k_i) C_{\curlywedge}(t_i)$$

For gravity

$$\mathcal{M} = N_{\curlywedge}(\epsilon_i, k_i) N_{\curlywedge}(\epsilon_i, k_i)\,,$$

but very nontrivial: graviton 3-vertex is much more complicated.

Can we extend to curved backgrounds?

- How do we define momentum eigenstates?
- What are momenta and polarization vectors?
- How can we relate Yang-Mills and Gravity?

Sandwich plane waves

The Brinkman form in d-dimensions of the metric is

$$ds^2 = dudv - Hdu^2 - dx_a dx^a\,, \quad a = 1, .., d-2\,.$$

with $H = H(u)_{ab} x^a x^b$, $H^a_a := R_{ab} l^a l^b = 0$ for vacuum.

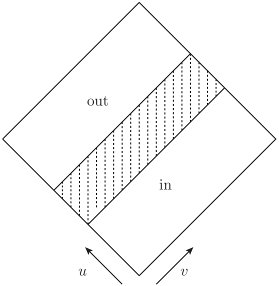

Figure: The sandwich plane wave with x^a-directions suppressed, $H_{ab}(u) \neq 0$ only in the shaded region with flat in- and out-regions.

- H_{ab} = curvature, supported for $u \in [0, 1]$ (shaded).
- These coordinates are global, but:
- Space-time *not* globally hyperbolic! (Penrose).

Plane wave symmetries: $2d-3$-Heisenberg group

$$ds^2 = dudv - H_{ab}(u)x^a x^b du^2 - dx_a dx^a$$

- Heisenberg group is transitive on $u = const.$, centre ∂_v.
- $2d-4$ killing vectors take form $e^a\partial_{x^a} - \dot{e}^a x^a \partial_v$ s.t.

$$\ddot{e}_a = H_{ab}e^b\,, \qquad \cdot = \frac{d}{du}$$

- Choose $d-2$-dimensional abelian subgroup

$$D_i = e^a_i \partial_{x^a} - \dot{e}^a_i x_a \partial_v, \qquad i = 1 \dots d-2\,,$$

  commuting $\Leftrightarrow \dot{e}^a_{[i} e_{j]a} = 0$.
- Let e^i_a be inverse matrix, $e^i_a e_{ib} = \delta_{ab}$.

Momentum eigenstates on plane waves: I. Gravity

- Choose $d-1$ commuting symmetries $(\partial_v, D_i) \rightsquigarrow$ Separable Hamilton-Jacobi soln, momenta (k_+, k_i)

$$\phi_k = k_+(v + \frac{1}{2}\sigma_{ab}x^a x^b) + k_i e^i_a x^a + \frac{k_i k_j F^{ij}(u)}{2k_+}\,,$$

where $F^{ij}(u) = \int^u e^i_a e^{ja} du'$ and $\sigma_{ab} = \dot{e}^i_a e_{bi}$ 'shear'.

- Then

$$\Phi_k = \frac{\mathrm{e}^{i\phi_k}}{\sqrt{|e|}}, \quad |e| = \det(e^a_i) \quad \text{solves} \qquad \Box\Phi_k = 0.$$

- Such a field has a 'curved' momentum

$$K_\mu dX^\mu := d\phi_k = k_+ dv + (\sigma_{ab}x^b + k_i e^i_a)dx^a + (\ldots)du$$

Memory:
As $u \to -\infty$, set $e^a_i = \delta^a_i$ so $K_\mu = (k_+, k_a, k_a k^a/2k_+)$ const.
As $u \to +\infty$, $e^a_i(u) = b^a_i + uc^a_i$, $b, c = const.$, and $\sigma_{ab} \neq 0$;
so wave fronts $\phi_k = const.$ become curved.

Higher spins

- We have $d-2$ covariantly constant spin raising operators

$$R^a = du\delta^{ab}\partial_{x^b} + dx^a\partial_v \,, \qquad \nabla_\mu R^a = 0 \,.$$

- Gives linear gauge field on background

$$A = \frac{\epsilon_a R^a}{k_+}\Phi_k = \varepsilon_\mu dX^\mu \Phi_k \,,$$

with curved polarization ε_μ, $K^\mu \varepsilon_\mu = 0$,

$$\varepsilon_\mu dX^\mu = \epsilon_a dx^a + \epsilon^a \left(\frac{k_i e^i_a}{k_+} + \sigma_{ab} x^b \right) du$$

- Linear gravity on background

$$h_{\mu\nu} dX^\mu dX^\nu = \frac{\epsilon_a R^a(\epsilon_b R^b \Phi_k)}{k_+^2} = \left((\varepsilon \cdot dX)^2 - \frac{i}{k_+}\epsilon_a \epsilon_b \sigma^{ab} du^2 \right) \Phi_k$$

Note potential obstruction to double copy.

Tails and Huygens

Theorem (Friedlander 1970s)

The only space-times that admit clean cut solutions to the wave equation are conformal to plane waves (or flat space).

- $\Phi = |e|^{-1/2}\delta(\phi_k)$ is clean cut solution to wave equation.
- Analogous spin-1 solution is

$$a = |e|^{-1/2}\epsilon_a R^a(\phi_k \Theta(\phi_k))$$

so

$$F = da = \delta(\phi_k)|e|^{-\frac{1}{2}}\epsilon_a R^a \phi_k \wedge d\phi_k + \Theta(\phi_k)|e|^{-\frac{1}{2}}\epsilon^a \sigma^0_{ab} dx^a \wedge du$$

i.e., there is backscattering with a tail.

- Similar spin-2 solution has longer tail.

Momentum eigenstates on plane waves: II. Yang-Mills

Use same coordinates on flat space-time with gauge potential

$$A = \dot{A}_a(u) x^a du \,, \qquad F = \dot{A}_a(u) dx^a \wedge du \,.$$

Again, take sandwich wave with $Supp(\dot{A}_a) \subset u \in [0, 1]$.
Momentum eigenstate charge e, $\Box_{eA}\Phi_k = 0$:

$$\Phi_k = \mathrm{e}^{i\left(k_+ v + (k_a + eA_a)x^a + \frac{f(u)}{2k_+}\right)} \,,$$

momentum $K_\mu(u) = \left(k_+, k_a + eA_a(u), \frac{\dot{f}(u)}{2k_+}\right)$ with

$$K \cdot K = 0 \quad \rightsquigarrow \quad f(u) = \int_{-\infty}^{u} (k_a + eA_a)(k^a + eA^a) du' \,.$$

Memory:
Choose $A_a = 0$ for $u < 0$, then for $u > 1$, $A_a(u) = const. \neq 0$.

$$K_\mu(u) = \begin{cases} \left(k_+, k_a, \frac{k_a k^a}{2k_+}\right) \,, & u < 0 \,, \\ \left(k_+, k_a + eA_a(1), \frac{\dot{f}(1)}{2k_+}\right) \,, & u > 1 \,. \end{cases}$$

Linear YM fields on the background

- $\dot{A}_a(u)x^a du$ valued in Cartan subalgebra $\mathfrak{h}$ of gauge Lie alg..
- Charged linear YM field a_μ satisfies

$$D^\mu D_{[\mu} a_{\nu]} + a^\mu \partial_{[\mu} e A_{\nu]} = 0\,, \qquad D_\mu = \partial_\mu + eA_\mu .$$

- Colour encoded in charge e =eigenvalue of $\mathfrak{h}\times$ coupling.
- Solution $a = \tilde{\epsilon}_a R^a \Phi_k = \tilde{\varepsilon}_\mu dX^\mu \Phi_k$, transverse polarization

$$\tilde{\varepsilon}_\mu(u) dX^\mu = \tilde{\epsilon}_a \left(dx^a + \frac{1}{k_+}(k^a + eA^a(u))du \right)\,, \quad \epsilon_a = const..$$

Convention: YM background polarization vectors are tilded.

No particle creation or leakage

As $u \to -\infty$ take linear fields to become flat space-time momentum eigenstates, i.e., $e^a_i = \delta^a_i$, and $A_a = 0$;

- $\pm$ frequency determined by sign of k_+, doesnt change with u so no particle creation.
- Inner products are u-independent on both backgrounds:

$$\langle \Phi_k | \Phi_{k'} \rangle = 2k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i) \, .$$

Similarly for spin-1

$$\langle a_1 | a_2 \rangle = 2\epsilon_1 \cdot \epsilon_2 k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i)$$

and spin-2

$$\langle h_1 | h_2 \rangle = 2(\epsilon_1 \cdot \epsilon_2)^2 k_+ \delta(k_+ - l_+) \delta^{d-2}(k_i - l_i) \, .$$

- Failure of global hyperbolicity does not lead to leakage.

[Failure in space of null geodesics: those parallel to ∂_v so co-dimension too high.]

Three particle gravity amplitude

- Cubic part of action give 3 vertex

$$\mathcal{M}_3 = \frac{\kappa}{4}\int d^d X\,(h_1^{\mu\nu}\partial_\mu h_{2\rho\sigma}\partial_\nu h_3^{\rho\sigma} - 2h_1^{\rho\nu}\partial_\mu h_{2\rho\sigma}\partial_\nu h_3^{\mu\sigma}) + perms$$

- Inserting our states yields

$$\frac{\kappa}{2}\delta^{d-1}\left(\sum_{r=1}^{3} k_r\right)\int \frac{du}{\sqrt{\det e_i^a}} \exp\left(\sum_{r=1}^{s} \frac{F^{ij}k_{ri}k_{rj}}{2k_{r0}}\right)$$
$$[(\varepsilon_1\cdot\varepsilon_2\,(K_1 - K_2)\cdot\varepsilon_3 + \circlearrowleft)^2 - ik_{1+}k_{2+}k_{3+}\sigma^{ab}\mathcal{C}_a\mathcal{C}_b]$$

where

$$\mathcal{C}_a := \varepsilon_1\cdot\varepsilon_2\frac{\epsilon_{3a}}{k_{3+}} + \circlearrowleft$$

- First term = (YM 3-pt amplitude)^2 on gravity background.
- However: tail term $\sigma^{ab}\mathcal{C}_a\mathcal{C}_b$ seems to obstruct double copy.

Three-point YM amplitude

- Cubic part of action $\int_M a_{[\mu} a_{\nu]} D^\mu a^\nu d^4X$ gives 3 point vertex

$$\mathcal{A}_3 = \int du \ \exp\left(i \sum_{r=1}^{3} \frac{f_r(u)}{2k_{r0}} \right) [\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2 \, \tilde{\varepsilon}_3 \cdot (K_1 - K_2) + \circlearrowleft] \, C_\curlywedge(t_i) \, .$$

- Bracketed term is in flat quantities

$$\left[\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \, \tilde{\epsilon}_3^a \left(\left(\frac{k_{1+}}{k_{2+}} k_{2a} - k_{1a} \right) + A_a \left(e_1 - e_2 \frac{k_{1+}}{k_{2+}} \right) \right) + \circlearrowleft \right] =: F + C.$$

- Second term gives background ‘tail’ dependence on A_a.

Double copy replacement principle

YM to GR uses replacement rules:

1. Flip charges $e_r \to -e_r$ so $\mathcal{A}_3 = F + C \to \tilde{\mathcal{A}} = F - C$ and
$$|\mathcal{A}_3|^2 := \mathcal{A}_3 \tilde{\mathcal{A}}_3 = F^2 - C^2$$
with $F = F(k_r, \tilde{\epsilon})$ and $C = C(k_r, \tilde{\epsilon}_r, A)$.
2. Replace $(k_{ra}, \tilde{\epsilon}_{ra})$ by $(k_i e^i_a, \epsilon_a) \rightsquigarrow F(k_{ri} e^i_a, \epsilon_r)$, $C(k_{ri} e^i_a, \epsilon_r, A)$.
3. Replace
$$e_r e_s A^a A^b \to \begin{cases} i k_{r0} \sigma^{ab} & r = s, \\ i(k_{r0} + k_{s0}) \sigma^{ab} & r \neq s\,. \end{cases}$$

Yields double copy of YM integrand for GR incorporating tails.

Four point amplitudes: YM

1810.05115 w/Adamo, Casali, Nekovar

- Construct scalar Feynman propagator as

$$G^F(X,X') = \int \frac{d^d k}{k^2 + i\epsilon} \exp i(\tilde{\phi}_k(X) - \tilde{\phi}_k(X'))$$

where $\tilde{\phi}_k$ is now solution to massive Hamilton-Jacobi

$$\tilde{\phi}_k = k_+ v + (\mathbf{k}_a + eA_a)x^a + \frac{1}{2k_+}\int^u ds[k^2 + (\mathbf{k} + eA(s))^2]$$

with $k^2 = k_+k_- - k_a k^a \neq 0$.

- For spin-1 solve $(\Box_{eA} + k^2)a_\mu + 2ieF_\mu^\nu a_\nu = 0$ with

$$G^F_{\mu\nu} = \int \frac{d^d k}{k^2 + i\epsilon} P_{\mu\nu}(u, u', k_+) \mathrm{e}^{i(\phi_k(X) - \phi_k(X'))}$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\delta_{ac} & \alpha\Delta A_a \\ 1 & -\alpha\Delta A_c & \frac{\alpha^2}{2}\Delta A^2 \end{pmatrix}, \quad \Delta A = A(u) - A(u'), \quad \alpha = \frac{ie}{k_+}$$

Gravity similar, but *much* more complicated.

BCJ at four points

YM diagrams are 3 exchanges in s, t and u channels and contact 4-vertex.

- BCJ form requires opening up 4-vertex into 3-exchange diagrams using $C_s \to sC_s/s$ etc..
- Then 3-channels, s, t and u in flat space give

$$\mathcal{A}_4 = \frac{N_s C_s}{s} + \frac{N_t C_t}{t} + \frac{N_u C_u}{u}$$

  with $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_1 + k_3)^2$.
- Jacobi-identity is $C_s - C_t + C_u = 0$; for BCJ property need

$$N_s - N_t + N_u = 0\,,$$

  but this is naively *obstructed* on a plane wave!
- With an additional mapping on scalar propagators, some nontrivial colour kinematics survives.

Conclusions and further developments

- Have explicit GR and YM Feynman rules on plane waves.
- Double copy is local in momentum space so nonlocal on space-time.
- Nontrivial double copy at 3pts, work in progress at 4pts, first steps towards nonlinear double copy in space-time.
- These formulae verify plane wave background ambitwistor string computation in arxiv:1708.09249.
- Ambitwistor strings manifest double copy. Full nonlinear version will give optimal nonlinear formulation?
- On Cosmological backgrounds?
  [Work in progress w/Tim Adamo and Atul Sharma].

Thank you!

Happy 60th birthday Jurek!