Elementary relational observables with quantum reference frames

In celebration of Jurek's 60th birthday

Chun-Yen Lin

Beijing Normal University

Quantum constraint kernel projector : $\hat{\mathbb{P}}$

- Quantum canonical GR : $\left.\begin{cases} \mathbb{K} \equiv Span\{\ |X_n\rangle\ \} \\ \{\hat{C}_\mu\} : \mathbb{K} \to \mathbb{K} \end{cases}\right\} \quad \delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}}$

-

Quantum constraint kernel projector : $\hat{\mathbb{P}}$

Quantum canonical GR : $\left\{ \begin{array}{l} \mathbb{K} \equiv Span\{\ |X_n\rangle\ \} \\ \{\hat{C}_\mu\} : \mathbb{K} \to \mathbb{K} \end{array} \right\} \quad \delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}}$

Perturb. Effective

QFT in FRW background

Non-perturb. Fund.

LQG

Quantum constraint kernel projector : $\hat{\mathbb{P}}$

Quantum canonical GR : $\left.\begin{cases} \mathbb{K} \equiv Span\{\ |X_n\rangle\ \} \\ \{\hat{C}_\mu\} : \mathbb{K} \to \mathbb{K} \end{cases}\right\} \delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}}$

Perturb. Effective

QFT in FRW background

$\langle X_n' | \hat{\mathbb{P}} | X_n \rangle$ = Faddeev-Popov path int (GR action)

$\hat{C}_\mu$: effective/renormalized, truncated

Local observables : refer to background spacetime

Non-perturb. Fund.

LQG

$\langle X_n' | \hat{\mathbb{P}} | X_n \rangle$ = well-defined operator components

$\hat{C}_\mu$: fundamental, with quantum geometry

Local observables : (?) abstract without background

Quantum constraint kernel projector : $\hat{\mathbb{P}}$

- Quantum canonical GR : $\left\{ \begin{array}{l} \mathbb{K} \equiv Span\{\ |X_n\rangle\ \} \\ \{\hat{C}_\mu\} : \mathbb{K} \to \mathbb{K} \end{array} \right\} \quad \delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}}$

Perturb. Effective QFT in FRW background	$\langle X_n' \vert \hat{\mathbb{P}} \vert X_n \rangle$ = Faddeev-Popov path int (GR action) $\hat{C}_\mu$: effective/renormalized, truncated Local observables : refer to background spacetime

Non-perturb. Fund. LQG	$\langle X_n' \vert \hat{\mathbb{P}} \vert X_n \rangle$ = well-defined operator components $\hat{C}_\mu$: fundamental, with quantum geometry Local observables : (?) abstract without background

- $\Rightarrow$ Goal : Universal local observables defined/computed using $\hat{\mathbb{P}}$

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_{\mu} \hat{C}_{\mu}^{2}\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^{*} \qquad \begin{cases} \mathbb{P} = \mathbb{P}^{\dagger} \\ \operatorname{diag} \mathbb{P} \geq 0 \end{cases}$$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^* \qquad \begin{cases} \mathbb{P} = \mathbb{P}^\dagger \\ \mathrm{diag}\,\mathbb{P} \geq 0 \end{cases}$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_{\mu} \hat{C}_{\mu}^{2}\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \rightarrow \mathbb{K}^{*} \quad \begin{cases} \mathbb{P} = \mathbb{P}^{\dagger} \\ \operatorname{diag} \mathbb{P} \geq 0 \end{cases}$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

$\mathbb{H}$ inner product : $\langle\psi_1|\,\hat{\mathbb{P}}\,|\psi_2\rangle \equiv (\Psi_1|\Psi_2)$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_{\mu}\hat{C}_{\mu}^{2}\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^{*} \qquad \begin{cases} \mathbb{P} = \mathbb{P}^{\dagger} \\ \operatorname{diag}\mathbb{P} \geq 0 \end{cases}$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

$\mathbb{H}$ inner product : $\langle\psi_1|\,\hat{\mathbb{P}}\,|\psi_2\rangle \equiv (\Psi_1|\Psi_2)$

Goal : $\mathcal{G}\left(\langle\psi_1|\,\hat{\mathbb{P}}\,|\psi_2\rangle\right) = (O(t_2)|O(t_1))$

[D.Marolf, J.Louko, D.Giulini, etc.]

Elementary Relational Observables in $\mathbb{H}$

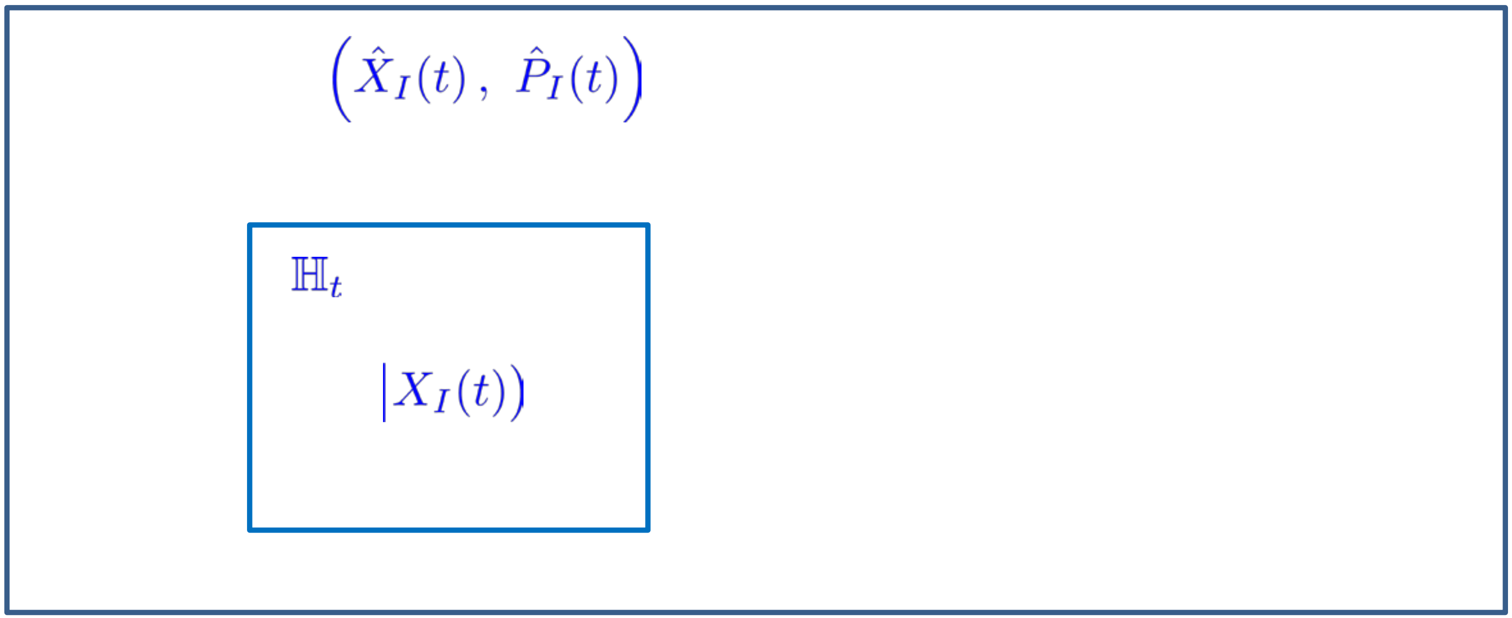
Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

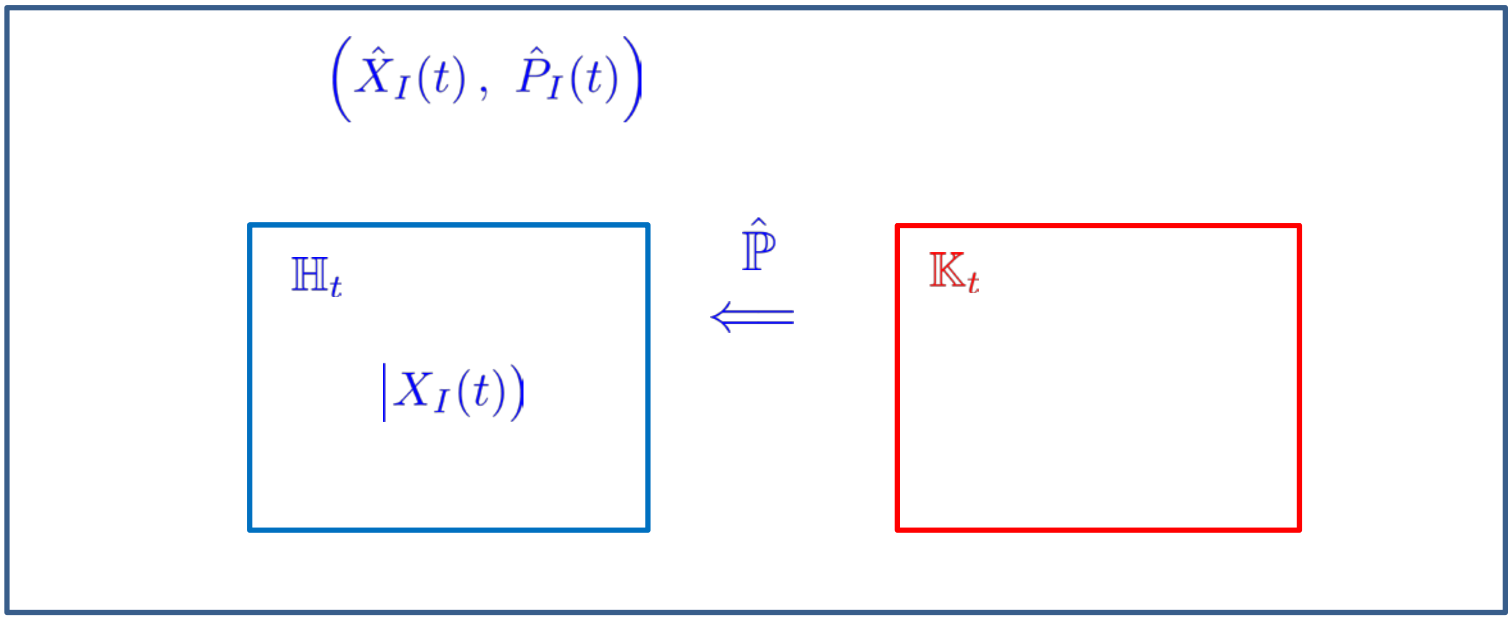
$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$



Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

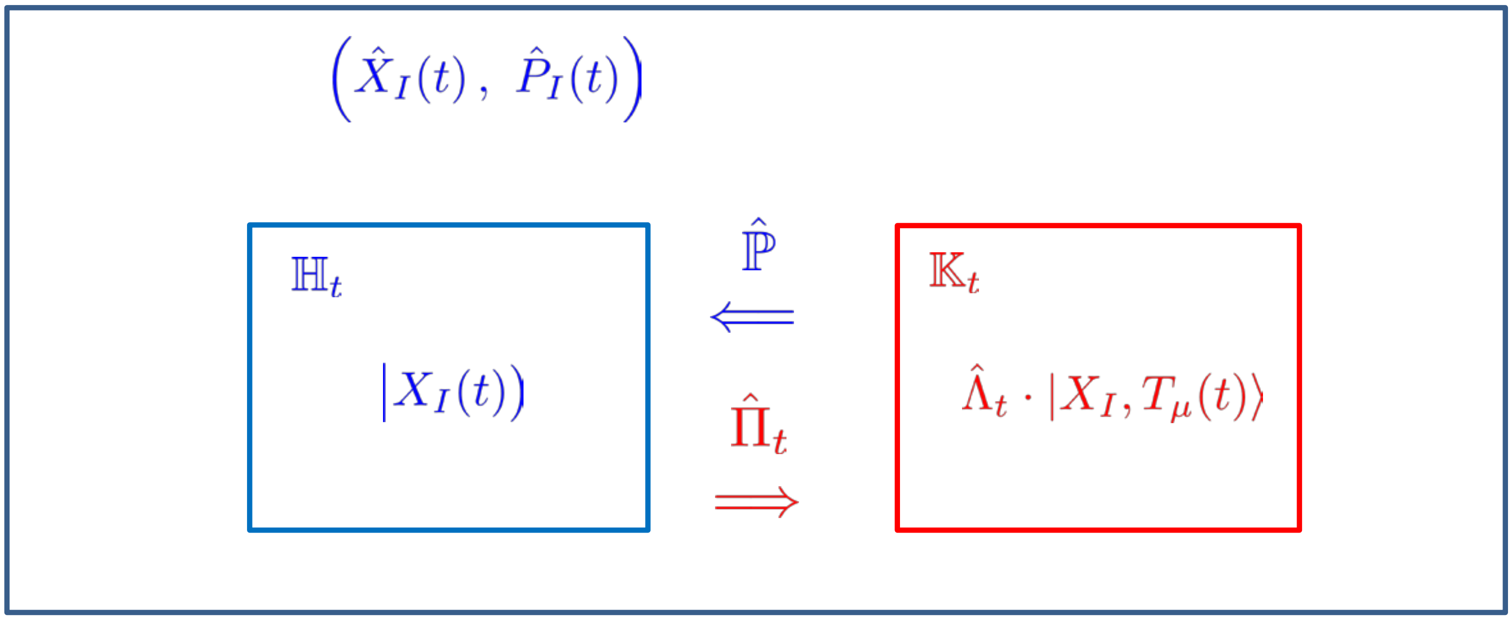
$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$



Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \;\leftrightarrow\; \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

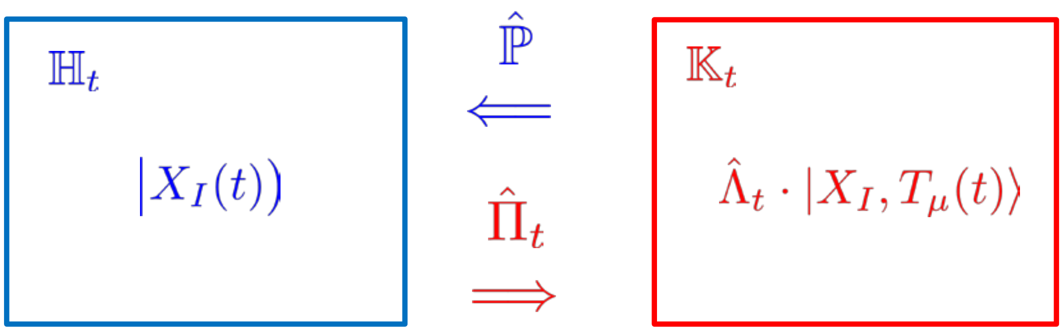


Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

$$\left(\hat{X}_I(t)\,,\ \hat{P}_I(t)\right) \quad \equiv \quad \hat{\mathbb{P}}\,\hat{\Lambda}_t \ \left(\hat{X}_I, \hat{P}_I\right) \ \hat{\Lambda}_t^{-1}\,\hat{\Pi}_t$$



$*$ Heisenberg Observables : Elementary Algebra , Quantum Covariance

$\mathcal{G}_{t_1,t_2}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle \langle X_I, T_\mu(t)| \\ \text{proper gauge } T_\mu(t) \iff \text{non-degenerate } (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}) : \mathbb{K} \to \mathbb{K} \end{cases}$$

$\mathcal{G}_{t_1,t_2}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\,\langle X_I, T_\mu(t)| \\ \text{proper gauge } T_\mu(t) \iff \text{non-degenerate } (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}) : \mathbb{K} \to \mathbb{K} \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\hat{t_2}\;\hat{\mathbb{P}}\;\hat{t_2})^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\hat{t_1}\;\hat{\mathbb{P}}\;\hat{t_1})^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$\mathcal{G}_{t_1,t_2}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle \langle X_I, T_\mu(t)| \\ \text{proper gauge } T_\mu(t) \iff \text{non-degenerate } (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}) : \mathbb{K} \to \mathbb{K} \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \left(\langle X_I', T_\mu(t_2)| \, (\hat{t_2}\,\hat{\mathbb{P}}\,\hat{t_2})^{-1/2} \right) \hat{\mathbb{P}} \left((\hat{t_1}\,\hat{\mathbb{P}}\,\hat{t_1})^{-1/2} \, |T_\mu(t_1), X_I\rangle \right)$$

$$\mathcal{G}_{t_2,t_1} = \left[(\mathbb{P}_{t_2t_2})^{-1/2} \; \mathbb{P}_{t_2t_1} \; (\mathbb{P}_{t_1t_1})^{-1/2} \right]$$

Relevant elements : $\mathbb{P}_{t_2t_1}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)| \; \hat{\mathbb{P}} \; |X_I, T_\mu(t_1)\rangle$

$\mathcal{G}_{t_1,t_2}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle \langle X_I, T_\mu(t)| \\ \text{proper gauge } T_\mu(t) \iff \text{non-degenerate } (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}) : \mathbb{K} \to \mathbb{K} \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\, (\hat{t_2}\,\hat{\mathbb{P}}\,\hat{t_2})^{-1/2} \Big)\, \hat{\mathbb{P}} \Big((\hat{t_1}\,\hat{\mathbb{P}}\,\hat{t_1})^{-1/2}\, |T_\mu(t_1), X_I\rangle \Big)$$

$$\mathcal{G}_{t_2,t_1} = \Big[(\mathbb{P}_{t_2t_2})^{-1/2}\; \mathbb{P}_{t_2t_1}\; (\mathbb{P}_{t_1t_1})^{-1/2} \Big]$$

Relevant elements : $\mathbb{P}_{t_2t_1}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\, \hat{\mathbb{P}}\, |X_I, T_\mu(t_1)\rangle$

When $\mathcal{G}_{t_2,t_1}$ converges :

(1) Exact relational observable : $\mathcal{G}_{t_2t_1}(X_I', X_I) = (X_I'(t_2)|X_I(t_1))$

$\mathcal{G}_{t_1,t_2}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\,\langle X_I, T_\mu(t)| \\ \text{proper gauge } T_\mu(t) \iff \text{non-degenerate } (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}) : \mathbb{K} \to \mathbb{K} \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\hat{t_2}\,\hat{\mathbb{P}}\,\hat{t_2})^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\hat{t_1}\,\hat{\mathbb{P}}\,\hat{t_1})^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$$\mathcal{G}_{t_2,t_1} = \Big[(\mathbb{P}_{t_2t_2})^{-1/2}\;\mathbb{P}_{t_2t_1}\;(\mathbb{P}_{t_1t_1})^{-1/2} \Big]$$

Relevant elements : $\mathbb{P}_{t_2t_1}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\;\hat{\mathbb{P}}\;|X_I, T_\mu(t_1)\rangle$

When $\mathcal{G}_{t_2,t_1}$ converges :

(1) Exact relational observable : $\mathcal{G}_{t_2t_1}(X_I', X_I) = (X_I'(t_2)|X_I(t_1))$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \;\Rightarrow\; \mathbb{H}_{t_2} = \mathbb{H}_{t_1} \equiv \mathbb{D}\;,\quad \Psi \in \mathbb{D} \to \Psi[X_I](t)$

Application : FRW LQC Perturbation $(v , \phi , \delta q_k^{(i)})$

$$\mathbb{K} = \mathbb{K}_{\text{LQC}}^{\text{FRW}} \otimes \mathbb{K}_{\text{Fock}}^{\delta} = Span\left\{ | v, \phi\rangle \otimes | N_k^{(1)}, N_k^{(2)}, N_k^{(3)}\rangle \right\}$$

Application : FRW LQC Perturbation $(v, \phi, \delta q_k^{(i)})$

$$\mathbb{K} = \mathbb{K}^{\rm FRW}_{\rm LQC} \otimes \mathbb{K}^{\delta}_{\rm Fock} = Span\left\{ |\, v,\, \phi\rangle \otimes |\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)}\rangle \right\}$$

$$\hat{C} = \hat{C}^{FRW}_{LQC} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta}\left(\hat{v},\, \widehat{\exp(i\sqrt{\Delta}\, l_p\, P_v)},\, \hat{\phi},\, \hat{P}_{\phi},\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, Thiemann, etc.]

Application : FRW LQC Perturbation $(v , \phi , \delta q_k^{(i)})$

$$\mathbb{K} = \mathbb{K}^{\mathrm{FRW}}_{\mathrm{LQC}} \otimes \mathbb{K}^{\delta}_{\mathrm{Fock}} = Span \left\{ |\, v,\, \phi\rangle \otimes |\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)}\rangle \right\}$$

$$\hat{C} = \hat{C}^{FRW}_{LQC} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta}\left(\hat{v},\, \exp(\widehat{i\sqrt{\Delta}\, l_p}\, P_v),\, \hat{\phi},\, \hat{P}_\phi,\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, Thiemann, etc.]

$$\hat{\mathbb{P}} \equiv \delta(\hat{C})$$

Application : FRW LQC Perturbation $(v , \phi , \delta q_k^{(i)})$

$$\mathbb{K} = \mathbb{K}^{\text{FRW}}_{\text{LQC}} \otimes \mathbb{K}^{\delta}_{\text{Fock}} = Span \left\{ |\, v,\, \phi\rangle \otimes |\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \right\}$$

$$\hat{C} = \hat{C}^{FRW}_{LQC} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta} \left(\hat{v},\, \exp(i \widehat{\sqrt{\Delta}\, l_p\, P_v}),\, \hat{\phi},\, \hat{P}_{\phi},\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, Thiemann, etc.]

$$\hat{\mathbb{P}} \equiv \delta(\hat{C})$$

$$T(t) \equiv \phi(t) \quad \Rightarrow \mathbb{K}_t = Span \left\{ |\, v,\, \phi(t) \,\rangle \otimes |\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \right\}$$

Relevant elements : $\mathbb{P}_{t_1 t_2} \left((v', N_k^{\prime (i)}) ,\; (v, N_k^{(i)}) \right)$

Application : FRW LQC Perturbation $(v \, , \, \phi \, , \, \delta q_k^{(i)} \,)$

$$\mathbb{K} = \mathbb{K}^{\rm FRW}_{\rm LQC} \otimes \mathbb{K}^{\delta}_{\rm Fock} = Span \left\{ \, | \, v, \, \phi \rangle \otimes | \, N_k^{(1)}, \, N_k^{(2)}, \, N_k^{(3)} \rangle \right\}$$

$$\hat{C} = \hat{C}^{FRW}_{LQC} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta} \left(\hat{v}, \, \widehat{\exp(i \sqrt{\Delta} \, l_p \, P_v)}, \, \hat{\phi}, \, \hat{P}_{\phi}, \, \hat{a}_k^{(i)}, \, \hat{a}_k^{(i)\dagger} \right)$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, Thiemann, etc.]

$$\hat{\mathbb{P}} \equiv \delta(\hat{C})$$

$$T(t) \equiv \phi(t) \quad \Rightarrow \mathbb{K}_t = Span \left\{ \, | \, v, \, \phi(t) \, \rangle \otimes | \, N_k^{(1)}, \, N_k^{(2)}, \, N_k^{(3)} \rangle \, \right\}$$

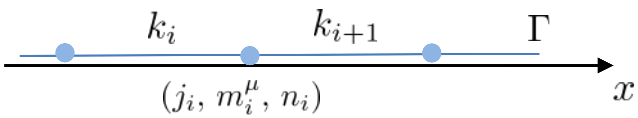
Relevant elements : $\mathbb{P}_{t_1 t_2} \left((v', N_k^{'(i)}) \, , \, (v, N_k^{(i)}) \right)$

$\Psi[\, v \, , \, N_k^{(i)} \,](t)$: full quantum interactions/back reactions

- Non-Gaussianity details in $O(\, \delta q^2 \,)$?
- Trans-Plankian $\delta q_k^{(i)}$: UV supression by quant geometry?

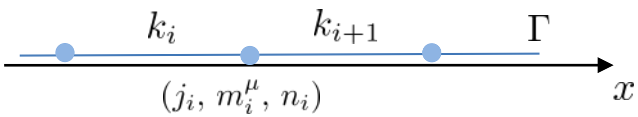
Application : Spherically Symm. LQG $(\, E_x(x)\,,\, E_\phi(x)\,,\, \Phi_{1,2}(x)\,,\, \psi(x)\,)$

$$\mathbb{K} = Span\{\ |\Gamma,\, k_i,\, j_i,\, m_i^{1,2},\, n_i\rangle\ \}$$



Application : Spherically Symm. LQG $(\, E_x(x)\,,\, E_\phi(x)\,,\, \Phi_{1,2}(x)\,,\, \psi(x)\,)$

$$\mathbb{K} = Span\{\ |\Gamma,\, k_i,\, j_i,\, m_i^{1,2},\, n_i\rangle\ \}$$

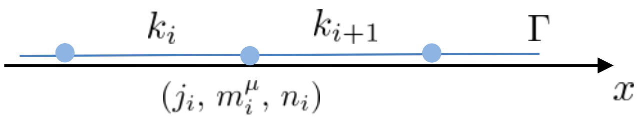


$$(\,\hat{C}_i\,,\,\hat{D}_i\,)\left(\widehat{F_x^i}\,,\,\widehat{h_x^i}\,,\,\widehat{F_\phi^i}\,,\,\widehat{h_\phi^i}\,,\,\hat{\Phi}_\mu^i\,,\,\hat{h}_{\Phi_\mu}^i\,,\,\hat{\psi}^i\,,\,\hat{h}_\psi^i\right)\quad(\,\Phi_{1,2}\,\text{massless}\,,\ \psi\,\text{massive}\,)$$

[M.Bojowald, R.Gambini, J.Pullin, J.Olmedo, L.Modesto, D.Chiou, C.Rovelli, A.Ashtekar, etc.]

Application : Spherically Symm. LQG $\left(E_x(x)\,,\, E_\phi(x)\,,\, \Phi_{1,2}(x)\,,\, \psi(x) \right)$

$$\mathbb{K} = Span\{ \ |\Gamma,\, k_i,\, j_i,\, m_i^{1,2},\, n_i\rangle \ \}$$



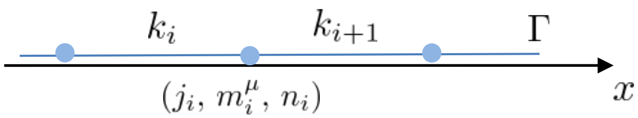
$$(\,\hat{C}_i\,,\,\hat{D}_i\,)\left(\widehat{F^i_x}\,,\,\widehat{h^i_x}\,,\,\widehat{F^i_\phi}\,,\,\widehat{h^i_\phi}\,,\,\hat{\Phi}^i_\mu\,,\,\hat{h}^i_{\Phi_\mu}\,,\,\hat{\psi}^i\,,\,\hat{h}^i_\psi\right)\quad(\,\Phi_{1,2}\,\text{massless}\,,\ \psi\,\text{massive}\,)$$

[M.Bojowald, R.Gambini, J.Pullin, J.Olmedo, L.Modesto, D.Chiou, C.Rovelli, A.Ashtekar, etc.]

$$\hat{M} \equiv \sum_i (\hat{C}_i)^2 + (\hat{D}_i)^2 \ \Rightarrow \ \hat{\mathbb{P}} \equiv \delta(\hat{M})$$

Application : Spherically Symm. LQG $\left(E_x(x) \,,\, E_\phi(x) \,,\, \Phi_{1,2}(x) \,,\, \psi(x) \right)$

$$\mathbb{K} = Span\{ \; |\Gamma,\, k_i,\, j_i,\, m_i^{1,2},\, n_i\rangle \; \}$$



$$(\hat{C}_i \,,\, \hat{D}_i) \left(\widehat{F_x^i} \,,\, \widehat{h_x^i} \,,\, \widehat{F_\phi^i} \,,\, \widehat{h_\phi^i} \,,\, \hat{\Phi}_\mu^i \,,\, \hat{h}_{\Phi_\mu}^i \,,\, \hat{\psi}^i \,,\, \hat{h}_\psi^i \right) \quad (\Phi_{1,2}\,\text{massless} \,,\; \psi\,\text{massive})$$

[M.Bojowald, R.Gambini, J.Pullin, J.Olmedo, L.Modesto, D.Chiou, C.Rovelli, A.Ashtekar, etc.]

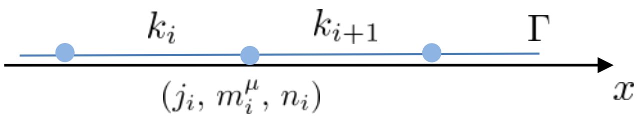
$$\hat{M} \equiv \sum_i (\hat{C}_i)^2 + (\hat{D}_i)^2 \;\Rightarrow\; \hat{\mathbb{P}} \equiv \delta(\hat{M})$$

$$(T_1(x,t) \,,\, T_2(x,t)) \equiv (\Phi_1(x,t) \,,\, \Phi_2(x,t)) \quad \Rightarrow \mathbb{K}_t = Span\{ \; |\Gamma,\, k_i,\, j_i,\, m_i^{1,2}(t),\, n_i\rangle \; \}$$

Relevant elements (spinfoam) : $\mathbb{P}_{t_1 t_2}\left((k_i',\, j_i',\, n_i') \,,\; (k_i,\, j_i,\, n_i) \right)$

Application : Spherically Symm. LQG $\left(E_x(x) \, , \, E_\phi(x) \, , \, \Phi_{1,2}(x) \, , \, \psi(x) \right)$

$$\mathbb{K} = Span\{ \; |\Gamma, \, k_i, \, j_i, \, m_i^{1,2}, \, n_i\rangle \; \}$$



$$(\, \hat{C}_i \, , \, \hat{D}_i \,) \left(\widehat{F_x^i} \, , \, \widehat{h_x^i} \, , \, \widehat{F_\phi^i} \, , \, \widehat{h_\phi^i} \, , \, \hat{\Phi}_\mu^i \, , \, \hat{h}_{\Phi_\mu}^i \, , \, \hat{\psi}^i \, , \, \hat{h}_\psi^i \right) \quad (\, \Phi_{1,2} \text{ massless} \, , \; \psi \text{ massive} \,)$$

[M.Bojowald, R.Gambini, J.Pullin, J.Olmedo, L.Modesto, D.Chiou, C.Rovelli, A.Ashtekar, etc.]

$$\hat{M} \equiv \sum_i (\hat{C}_i)^2 + (\hat{D}_i)^2 \;\; \Rightarrow \;\; \hat{\mathbb{P}} \equiv \delta(\hat{M})$$

$$(\, T_1(x,t) \, , \, T_2(x,t) \,) \equiv (\, \Phi_1(x,t) \, , \, \Phi_2(x,t)) \quad \Rightarrow \mathbb{K}_t = Span\{ \; |\Gamma, \, k_i, \, j_i, \, m_i^{1,2}(t), \, n_i\rangle \; \}$$

Relevant elements (spinfoam) : $\mathbb{P}_{t_1 t_2} \left(\, (k_i', \, j_i', \, n_i') \, , \; (k_i, \, j_i, \, n_i) \, \right)$

$\Psi[k_i, \, j_i, \, n_i](t)$: full interactions/ full Cauchy surface

- Very massive ψ : LQG collapse
- Whole quantum spacetime BH evolution

Conclusion and Outlook

N-point functions: $\hat{O}(\hat{X}_I, \hat{P}_I)$

$$(X_I'(t_0) | \hat{O}_2(t_2)\ \hat{O}_1(t_1)\ | X_I(t_0)\ ) = [\, U_{t_0 t_2}\ O_2\ U_{t_2 t_1}\ O_1\ U_{t_1 t_0}\,]\ (X_I', X_I)$$

Existing applications:

* FRW LQC with massless ϕ [C.Y.L., J.Lewandowski]

* Conformal Brans-Dicke Bianchi I LQC [C.Y.L., Y.Ma]

Ongoing works:

* FRW LQC perturbation with inflaton ϕ

* Schwarzchild LQC : with massless $\phi_{\mu=1,2}(r)$ and very massive $\Phi(r)$

$\Rightarrow$ LQG collapse with 1-D spinfoam

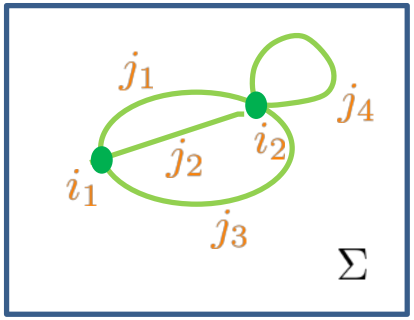
Loop Quantum Gravity

GR in Ashtekar formulation $\left\{ \begin{array}{l} q \;\rightarrow\; E_{SU(2)} \;\rightarrow\; F_s(E) \\ K \;\rightarrow\; A_{SU(2)} \;\rightarrow\; h_e^{(j)}(A) \end{array} \right.$

[A.Ashtekar, L.Smolin, C.Rovelli, J.Lewandowski, T.Thiemann, J.Baez, A.Perez, J.Engle, R.Pereira, E.Livine, L.Freidel, etc.]

Loop Quantum Gravity

GR in Ashtekar formulation $\begin{cases} q \to E_{SU(2)} \to F_s(E) \\ K \to A_{SU(2)} \to h_e^{(j)}(A) \end{cases}$

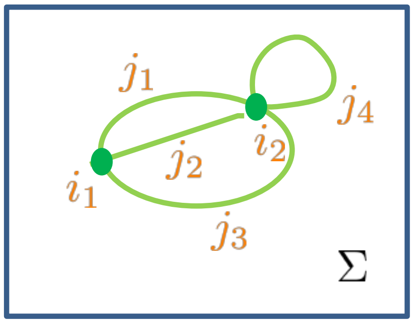
$\psi_{\Gamma, c}[A] =$ 

[A.Ashtekar, L.Smolin, C.Rovelli, J.Lewandowski, T.Thiemann, J.Baez, A.Perez, J.Engle, R.Pereira, E.Livine, L.Freidel, etc.]

Loop Quantum Gravity

GR in Ashtekar formulation
$$\begin{cases} q \rightarrow E_{SU(2)} \rightarrow F_s(E) \\ K \rightarrow A_{SU(2)} \rightarrow h_e^{(j)}(A) \end{cases}$$

$\psi_{\Gamma,\,c}\,[A] =$



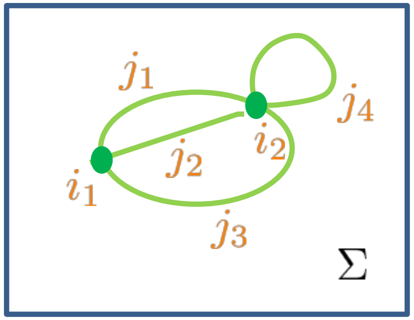
$$\begin{cases} \widehat{\text{Area}}(\,\hat{F}_s\,)\;|\psi_{\Gamma,\,c}\rangle = N_c \cdot l_p^2\;\;|\psi_{\Gamma,\,c}\rangle \\ \widehat{\text{R}}(\,\hat{h}_e^{(j)}\,)\;|\psi_{\Gamma,\,c}\rangle = \sum f_{\Gamma',\,c'}\;\;|\psi_{\Gamma',\,c'}\rangle \end{cases}$$

[A.Ashtekar, L.Smolin, C.Rovelli, J.Lewandowski, T.Thiemann, J.Baez, A.Perez, J.Engle, R.Pereira, E.Livine, L.Freidel, etc.]

Loop Quantum Gravity

GR in Ashtekar formulation $\begin{cases} q \to E_{SU(2)} \to F_s(E) \\ K \to A_{SU(2)} \to h_e^{(j)}(A) \end{cases}$

$\psi_{\Gamma,\,c}[A] =$



$$\begin{cases} \widehat{\text{Area}}(\,\hat{F}_s\,)\,|\psi_{\Gamma,\,c}\rangle = N_c \cdot l_p^2\;|\psi_{\Gamma,\,c}\rangle \\ \hat{\text{R}}(\,\hat{h}_e^{(j)}\,)\,|\psi_{\Gamma,\,c}\rangle = \sum f_{\Gamma',\,c'}\;|\psi_{\Gamma',\,c'}\rangle \end{cases}$$

$$\begin{cases} \mathbb{K} \equiv Span\{\,|\,\psi_{\Gamma,\,c}\rangle\,\} \\ \hat{\mathbb{P}} \equiv \delta\Big(\,\hat{C}_0(\,\hat{F}_s,\,\hat{h}_e^{(j)},\,\hat{\phi},\,\hat{h}_\phi\,)\,\Big)\ \text{well-defined, computable (spin-foam)} \end{cases}$$

[A.Ashtekar, L.Smolin, C.Rovelli, J.Lewandowski, T.Thiemann, J.Baez, A.Perez, J.Engle, R.Pereira, E.Livine, L.Freidel, etc.]

FRW LQC Perturbation with Inflaton ϕ

$$\mathbb{K} \equiv Span\left\{ |\, v,\, \phi,\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \;;\; N_k^{(i)} = \delta q_k^{(i)} \text{ Fock exitation} \right\}$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, etc.]

FRW LQC Perturbation with Inflaton ϕ

$$\mathbb{K} \equiv Span\left\{ |\, v,\, \phi,\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \;;\; N_k^{(i)} = \delta q_k^{(i)} \text{ Fock exitation} \right\}$$

$$\hat{C} = \hat{C}_{LQC}^{FRW} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta}\left(\hat{v},\, \widehat{\exp(i\sqrt{\Delta}\, l_p\, P_v)},\, \hat{\phi},\, \hat{P}_\phi,\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán,
Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, etc.]

FRW LQC Perturbation with Inflaton ϕ

$$\mathbb{K} \equiv Span\left\{ |\, v,\, \phi,\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \;;\; N_k^{(i)} = \delta q_k^{(i)} \text{ Fock exitation} \right\}$$

$$\hat{C} = \hat{C}^{FRW}_{LQC} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta}\left(\hat{v},\, \exp(i\,\widehat{\sqrt{\Delta}\, l_p}\, P_v),\, \hat{\phi},\, \hat{P}_\phi,\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

$\text{Approx Dynamics: Test field on dressed metric} < \hat{v} > \Rightarrow \text{approx Schrödinger } \widehat{\delta q}_k^{(i)}(t)$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán,
Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, etc.]

FRW LQC Perturbation with Inflaton ϕ

$$\mathbb{K} \equiv Span\left\{ |\, v,\, \phi,\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)}\rangle \;;\; N_k^{(i)} = \delta q_k^{(i)} \text{ Fock exitation} \right\}$$

$$\hat{C} = \hat{C}_{LQC}^{FRW} + \hat{C}^{V}(\hat{v}, \hat{\phi}) + \hat{C}^{\delta}\left(\hat{v},\, \exp(i\,\widehat{\sqrt{\Delta}\, l_p}\, P_v),\, \hat{\phi},\, \hat{P}_{\phi},\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

Approx Dynamics: Test field on dressed metric $< \hat{v} > \Rightarrow$ approx Schrödinger $\widehat{\delta q}_k^{(i)}(t)$

- Quantum back reaction competing in Non-Gaussianity details in $O(\,\delta q^2\,)$?
- Trans-Plankian $\delta q_k^{(i)}$: UV supression by quant geometry?

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, etc.]

FRW LQC Perturbation with Inflaton ϕ

$$\mathbb{K} \equiv Span\left\{ |\, v,\, \phi,\, N_k^{(1)},\, N_k^{(2)},\, N_k^{(3)} \rangle \;;\; N_k^{(i)} = \delta q_k^{(i)} \text{ Fock exitation} \right\}$$

$$\hat{C} = \hat{C}_{LQC}^{FRW} + \hat{C}^V(\hat{v}, \hat{\phi}) + \hat{C}^\delta \left(\hat{v},\, \exp(i\, \widehat{\sqrt{\Delta}\, l_p\, P_v}),\, \hat{\phi},\, \hat{P}_\phi,\, \hat{a}_k^{(i)},\, \hat{a}_k^{(i)\dagger} \right)$$

Approx Dynamics : Test field on dressed metric $< \hat{v} > \Rightarrow$ approx Schrödinger $\widehat{\delta q}_k^{(i)}(t)$

- Quantum back reaction competing in Non-Gaussianity details in $O(\,\delta q^2\,)$?
- Trans-Plankian $\delta q_k^{(i)}$: UV supression by quant geometry?

$$\text{Goal}: \; (\Psi' |\, \widehat{\delta q}_k^{(i)}(t_2)\, \widehat{\delta q}_{k'}^{(i)}(t_1)\, |\Psi) = \mathcal{F}(\mathbb{P}) \;\; \text{with } \hat{\mathbb{P}} = \delta(\hat{C})$$

[M.Bojowald, A.Ashtekar, B.Navascués, M.Martín-Benito, G.A.Mena Marugán, Q.Wu, T.Zhu, A.Wang, I.Agullo, B.Bolliet, V.Sreenath, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^* \quad \left\{ \begin{array}{l} \mathbb{P} = \mathbb{P}^\dagger \\ \operatorname{diag} \mathbb{P} \geq 0 \end{array} \right.$$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_\mu \hat{C}_\mu^2\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^* \quad \left\{ \begin{array}{l} \mathbb{P} = \mathbb{P}^\dagger \\ \operatorname{diag} \mathbb{P} \geq 0 \end{array} \right.$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_{\mu} \hat{C}_{\mu}^{2}\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^{*} \qquad \left\{ \begin{array}{l} \mathbb{P} = \mathbb{P}^{\dagger} \\ \mathrm{diag}\, \mathbb{P} \geq 0 \end{array} \right.$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

$\mathbb{H}$ inner product : $\langle\psi_1|\,\hat{\mathbb{P}}\,|\psi_2\rangle \equiv (\Psi_1|\Psi_2)$

[D.Marolf, J.Louko, D.Giulini, etc.]

Physical Hilbert Space from $\hat{\mathbb{P}}$

$$\delta\left(\sum_{\mu} \hat{C}_{\mu}^{2}\right) \equiv \hat{\mathbb{P}} : \mathbb{K} \to \mathbb{K}^{*} \quad \begin{cases} \mathbb{P} = \mathbb{P}^{\dagger} \\ \operatorname{diag}\mathbb{P} \geq 0 \end{cases}$$

Refined Algebraic Quant $\quad \hat{\mathbb{P}} : \mathbb{K} \longrightarrow \mathbb{H}$

$$|\psi_1\rangle \mapsto |\Psi_1)$$

$\mathbb{H}$ inner product : $\langle\psi_1|\,\hat{\mathbb{P}}\,|\psi_2\rangle \equiv (\Psi_1|\Psi_2)$

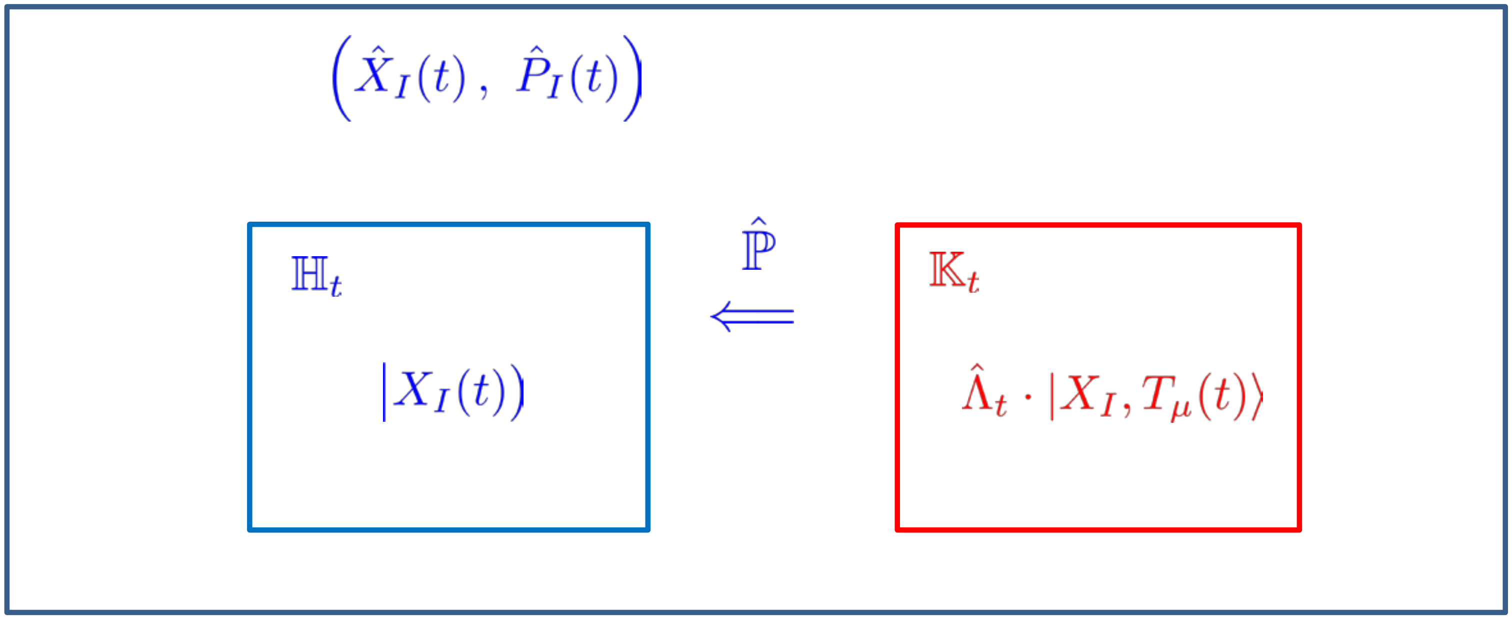
Goal : $(O(t_2)|O(t_1)) = \mathcal{G}(\mathbb{P})$

[D.Marolf, J.Louko, D.Giulini, etc.]

Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \;\leftrightarrow\; \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$



Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

$$\left(\hat{X}_I(t)\,,\ \hat{P}_I(t)\right) \equiv \hat{\mathbb{P}}\,\hat{\Lambda}\,\left(\hat{X}_I, \hat{P}_I\right)\,\hat{\Lambda}^{-1}\,\hat{\Pi}_t$$

$\mathbb{H}_t$

$$\big|X_I(t)\big)$$

$$\xleftarrow{\hat{\mathbb{P}}}$$

$$\xrightarrow{\hat{\Pi}_t}$$

$\mathbb{K}_t$

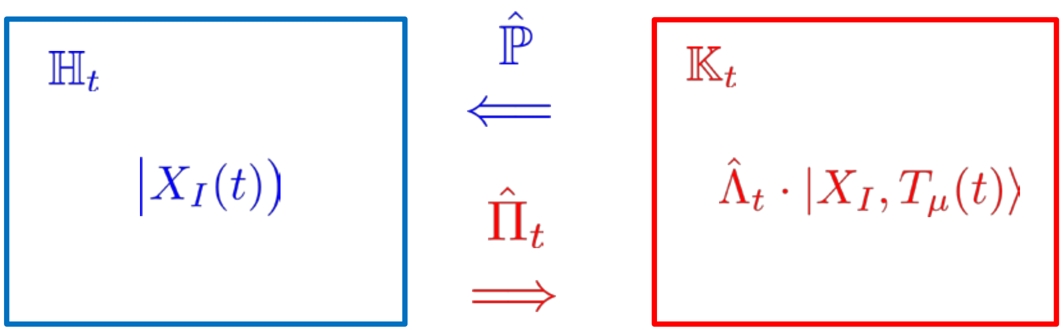
$$\hat{\Lambda}_t \cdot |X_I, T_\mu(t)\rangle$$

Elementary Relational Observables in $\mathbb{H}$

Relational description with $\mathbb{K} \equiv Span\{|X_I, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t) \leftrightarrow \mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

$$\left(\hat{X}_I(t)\,,\ \hat{P}_I(t)\right) \quad \equiv \quad \hat{\mathbb{P}}\,\hat{\Lambda}\ \left(\hat{X}_I, \hat{P}_I\right)\ \hat{\Lambda}^{-1}\,\hat{\Pi}_t$$



$*$ Heisenberg Observables : Elementary Algebra , Quantum Covariance

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}\,)^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\,\langle X_I, T_\mu(t)| \ : \ \text{Quantum Reference Frame} \\ \text{diag}(\ \hat{t}\ \hat{\mathbb{P}}\ \hat{t}\ ) > 0 \ \text{ with proper gauge } T_\mu(t) \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\,\hat{t_2}\ \hat{\mathbb{P}}\ \hat{t_2}\,)^{-1/2}\Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\ \hat{\mathbb{P}}\ \hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle\Big)$$

$$= \Big[(\mathbb{P}_{t_2 t_2})^{-1/2}\ \mathbb{P}_{t_2 t_1}\ (\mathbb{P}_{t_1 t_1})^{-1/2}\Big]\,(X_I', X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1 t_2}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\ \hat{\mathbb{P}}\ |X_I, T_\mu(t_1)\rangle$

When convergent :

(1) Exact relational observable : $(X_I'(t_1)|X_I(t_1)) = I(X_I', X_I)$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}$, $\Psi \in \mathbb{D} \to \Psi[X_I](t)$

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\langle X_I, T_\mu(t)| \ : \ \text{Quantum Reference Frame} \\ \text{diag}(\ \hat{t}\,\hat{\mathbb{P}}\,\hat{t}\ ) > 0 \ \text{ with proper gauge } T_\mu(t) \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\,\hat{t_2}\ \hat{\mathbb{P}}\ \hat{t_2}\,)^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\ \hat{\mathbb{P}}\ \hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$$= \Big[(\mathbb{P}_{t_2 t_2})^{-1/2}\ \mathbb{P}_{t_2 t_1}\ (\mathbb{P}_{t_1 t_1})^{-1/2} \Big]\,(X_I', X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1 t_2}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\ \hat{\mathbb{P}}\ |X_I, T_\mu(t_1)\rangle$

When convergent :

(1) Exact relational observable : $(X_I'(t_1)|X_I(t_1)) = I(X_I', X_I)$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}$, $\Psi \in \mathbb{D} \to \Psi[X_I](t)$

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t}\,)^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\,\langle X_I, T_\mu(t)| \ : \ \text{Quantum Reference Frame} \\ \text{diag}(\ \hat{t}\ \hat{\mathbb{P}}\ \hat{t}\ ) > 0 \ \text{with proper gauge } T_\mu(t) \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\,\hat{t_2}\ \hat{\mathbb{P}}\ \hat{t_2}\,)^{-1/2}\Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\ \hat{\mathbb{P}}\ \hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle\Big)$$

$$= \Big[(\mathbb{P}_{t_2t_2})^{-1/2}\ \mathbb{P}_{t_2t_1}\ (\mathbb{P}_{t_1t_1})^{-1/2}\Big]\,(X_I', X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1t_2}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\ \hat{\mathbb{P}}\ |X_I, T_\mu(t_1)\rangle$

When convergent :

(1) Exact relational observable : $(X_I'(t_1)|X_I(t_1)) = I(X_I', X_I)$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}\ ,\quad \Psi \in \mathbb{D} \to \Psi[X_I](t)$

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \widehat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle \langle X_I, T_\mu(t)| \ : \ \text{Quantum Reference Frame} \\ \text{diag}(\ \hat{t}\,\hat{\mathbb{P}}\,\hat{t}\ ) > 0 \ \text{ with proper gauge } T_\mu(t) \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\,\hat{t_2}\,\hat{\mathbb{P}}\,\hat{t_2}\,)^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\,\hat{\mathbb{P}}\,\hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$$= \left[(\mathbb{P}_{t_2 t_2})^{-1/2}\ \mathbb{P}_{t_2 t_1}\ (\mathbb{P}_{t_1 t_1})^{-1/2} \right] (X_I', X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1 t_2}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\ \hat{\mathbb{P}}\ |X_I, T_\mu(t_1)\rangle$

When convergent :

(1) Exact relational observable : $(X_I'(t_1)|X_I(t_1)) = I(X_I', X_I)$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}\ ,\quad \Psi \in \mathbb{D} \to \Psi[X_I](t)$

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \widehat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle \langle X_I, T_\mu(t)| \; : \text{ Quantum Reference Frame} \\ \text{diag}(\,\hat{t}\,\hat{\mathbb{P}}\,\hat{t}\,) > 0 \text{ with proper gauge } T_\mu(t) \end{cases}$$

$$(X_I'(t_2)|X_I(t_1)) = \Big(\langle X_I', T_\mu(t_2)|\,(\,\hat{t_2}\;\hat{\mathbb{P}}\;\hat{t_2}\,)^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\;\hat{\mathbb{P}}\;\hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$$= \left[(\mathbb{P}_{t_2 t_2})^{-1/2}\;\mathbb{P}_{t_2 t_1}\;(\mathbb{P}_{t_1 t_1})^{-1/2} \right] (X_I', X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1 t_2}(X_I', X_I) \equiv \langle X_I', T_\mu(t_2)|\;\hat{\mathbb{P}}\;|X_I, T_\mu(t_1)\rangle$

When convergent :

(1) Exact relational observable : $(X_I'(t_1)|X_I(t_1)) = I(X_I', X_I)$

(2) When $(X_I'(t_2)|X_I(t_1)) = U_{t_2,t_1}(X_I', X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}$, $\Psi \in \mathbb{D} \to \Psi[X_I](t)$

Function $\mathcal{G}(\mathbb{P})$ for Propagator

$$\hat{\Lambda}_t = (\hat{t}\,\hat{\mathbb{P}}\,\hat{t})^{-1/2} \quad \begin{cases} \hat{t} \equiv \sum_{X_I} |X_I, T_\mu(t)\rangle\langle X_I, T_\mu(t)| \ : \ \text{Quantum Reference Frame} \\ \text{diag}(\,\hat{t}\,\hat{\mathbb{P}}\,\hat{t}\,) > 0 \ \text{ with proper gauge } T_\mu(t) \end{cases}$$

$$(X'_I(t_2)|X_I(t_1)) = \Big(\langle X'_I, T_\mu(t_2)|\,(\,\hat{t_2}\,\hat{\mathbb{P}}\,\hat{t_2}\,)^{-1/2} \Big)\,\hat{\mathbb{P}}\,\Big((\,\hat{t_1}\,\hat{\mathbb{P}}\,\hat{t_1}\,)^{-1/2}\,|T_\mu(t_1), X_I\rangle \Big)$$

$$= \Big[(\mathbb{P}_{t_2 t_2})^{-1/2}\ \mathbb{P}_{t_2 t_1}\ (\mathbb{P}_{t_1 t_1})^{-1/2} \Big]\,(X'_I, X_I)$$

Relevant amplitudes : $\mathbb{P}_{t_1 t_2}(X'_I, X_I) \equiv \langle X'_I, T_\mu(t_2)|\ \hat{\mathbb{P}}\ |X_I, T_\mu(t_1)\rangle$

When convergent :

$$\begin{cases} (1)\ \text{Exact relational observable} :\ (X'_I(t_1)|X_I(t_1)) = I(X'_I, X_I) \\ (2)\ \text{When } (X'_I(t_2)|X_I(t_1)) = U_{t_2,t_1}(X'_I, X_I) \Rightarrow \mathbb{H}_{t'} = \mathbb{H}_t \equiv \mathbb{D}\ ,\quad \Psi \in \mathbb{D} \to \Psi[X_I](t) \end{cases}$$

Application and Outlook

N-point functions: $\hat{O}(\hat{X}_I, \hat{P}_I)$

$$(X_I'(t_0) |\, \hat{O}_2(t_2)\ \hat{O}_1(t_1)\, | X_I(t_0)) = [\, U_{t_0 t_2}\ O_2\ U_{t_2 t_1}\ O_1\ U_{t_1 t_0} \,]\ (X_I', X_I)$$

Existing applications:

* FRW LQC with massless ϕ [C.Y.L., J.Lewandowski]
* Conformal Brans-Dicke Bianchi I LQC [C.Y.L., Y.Ma]

Ongoing works:

* FRW LQC perturbation with inflaton ϕ
* Schwarzchild LQC : with massless $\phi_{\mu=1,2}(r)$ and very massive $\Phi(r)$

$\Rightarrow$ LQG collapse with 1-D spinfoam

Application and Outlook

Existing Applications :

* FRW LQC with massless ϕ

C.Y.L., Jerzy Lewandowski, Phys. Rev. D 98, 026023 (2018)

* Conformal Brans-Dicke Bianchi I LQC

under preparation, see Yongge Ma's talk

Next Challenge :

* Spherical Symm. LQG

$\Rightarrow$ Full Cauchy Slicing, Back Reaction, Loop Correction

E-H Path Integral Formulation

- Relational description with $\mathbb{K} \equiv Span\{|X_n, X_\mu\rangle\}$

$*$ Choose $\hat{T}_\mu = \hat{T}_\mu(\hat{X}_\mu, \hat{P}_\mu)$ and $T_\mu(t)$ $\rightarrow$ eigenspace $\mathbb{K}_t \equiv Span\{|X_I, T_\mu(t)\rangle\}$

$*$ Relevant amplitude matrix $\mathbb{P}_{t',t}(X'_I, X_I) \equiv \langle X'_I, T_\mu(t')|\; \hat{\mathbb{P}}\; |T_\mu(t), X_I\rangle$

- $\hat{\mathbb{P}} : \mathbb{K}_t \rightarrow \mathbb{H}_t$ Physical inner product : $\mathbb{P}_{t=t}$

$* \mathbb{P}_{t=t} = \mathbb{P}^\dagger_{t=t}$ and $\mathbb{P}_{t=t}(A, A) \geq 0$

$*$ Assumption : $\mathbb{P}_{t,t}$ real-diagonalizable

- $diag\,(\mathbb{P}_{t,t}) \ni 0 \Leftrightarrow T_\mu(t)$ too weak
- $diag\,(\mathbb{P}_{t,t}) > 0 \Leftrightarrow$ Isometry $\mathbb{K}_t \equiv \mathbb{H}_t$

- $diag\,(\mathbb{P}_{t,t}) > 0$; We can introduce $\hat{\Lambda} : \mathbb{K}_t \rightarrow \mathbb{K}_t$ with $\Lambda_t = (\mathbb{P}_{t,t})^{-1/2}$

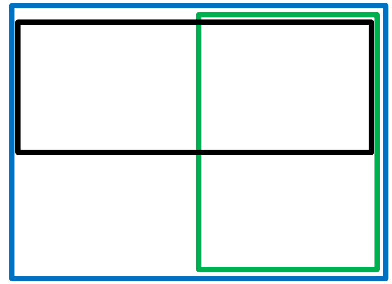
$$* \; (\mathbb{P}_{tt})^{-1/2}\, \mathbb{P}_{tt}\, (\mathbb{P}_{tt})^{-1/2}(X'_I, X_I) = \langle X'_I, T_\mu(t)|\, \hat{\Lambda}^\dagger\, \hat{\mathbb{P}}\, \hat{\Lambda}\, |T_\mu(t), X_I\rangle$$

$$= \langle X'_I | X_I \rangle \qquad\qquad = (\, X'_I(t) \,|\, X_I(t)\,)$$

$*$ Isometry $(\hat{\mathbb{P}} \cdot \hat{\Lambda}) : \mathbb{K}_t \rightarrow \mathbb{H}_t$; with $(\hat{\mathbb{P}} \cdot \hat{\Lambda})\,|X_I, T_\mu(t)\rangle \equiv |\, X_I(t)\,)$

Flat FRW LQC + massless Klien-Gordon Scalar Field .

$\mathbb{D} = Span\{\,|+, P_\phi)\}$ $\qquad$ $\tilde{\mathbb{D}} = Span\{\,|\Omega, +)\}$



$\left(|\mathbf{V}\rangle = \frac{|V\rangle + |V + \bar{v}\rangle}{\sqrt{2}} \right)$

$\hat{T} = \theta(\hat{\Omega})\,\hat{\mathbf{V}}\ ;\ \ T(\tau) = \tau$

$\hat{T} = \theta(\hat{P}_\phi)\,\hat{\phi}\ ;\ \ T(t) = t$

$\langle T(\tau'), \phi'|\ \hat{\mathbb{P}}\ |T(\tau), \phi\rangle$

$\mathbb{H}$

$\langle V', T(t')|\ \hat{\mathbb{P}}\ |V, T(t)\rangle$

$\Longrightarrow \begin{cases} (1)\ \hat{\Lambda}\ \text{found} \\ (2)\ \hat{U}\ \text{verified} \end{cases}$

$\Longrightarrow \begin{cases} (1)\ \hat{\Lambda}\ \text{found} \\ (2)\ \hat{U}\ \text{verified} \end{cases}$

$$\Longrightarrow \begin{cases} \Psi_{\mathbb{D}}[\phi](\tau) \\ \hat{\phi}(\tau), \hat{P}_\phi(\tau)\ :\ \mathbb{D} \to \mathbb{D} \\ \hat{H}_{lKG} \underset{\bar{\mu}\to 0}{\underset{\lim}{\longrightarrow}} \hat{H}_{KG} \end{cases}$$

$$\Longrightarrow \begin{cases} \Psi_{\tilde{\mathbb{D}}}[V](t) \\ \hat{V}(t), \hat{\Omega}(t)\ :\ \tilde{\mathbb{D}} \to \tilde{\mathbb{D}} \\ \hat{H}_{lqc} \underset{\bar{\mu}\to 0}{\underset{\lim}{\longrightarrow}} \hat{H}_{FRW} \end{cases}$$

Flat FRW LQC + massless Klien-Gordon Scalar Field

- $\mathbb{K} \equiv Span\{\, |V_n, P_\phi\rangle \;;\; V_n = (1+2n)\bar{v} \,\}$; inner product $\delta_{V',V}\, \delta(P'_\phi - P_\phi)$

$$\begin{cases} \text{Gravity} : \; [\,\hat{V}\,,\, \widehat{e^{-i\bar{\mu}c}}\,] = \bar{v}\; \widehat{e^{-i\bar{\mu}c}} \;\longrightarrow\; \widehat{e^{-i\bar{\mu}c}}\; |V,\, P_\phi\rangle = |V+\bar{v},\, P_\phi\rangle \\ \qquad\qquad (\;\bar{\mu} \sim l_p \;\;;\;\; \bar{v} \sim l_p^3\;) \text{ quantum geometry} \\ \text{K.G. Scalar} : \; [\,\hat{\phi}\,,\, \hat{P}_\phi\,] = \hbar \end{cases}$$

- $\hat{C}_0 \,\equiv -\dfrac{6}{\gamma^2}\,\hat{\Omega}^2 + 8\pi G\,\hat{P}_\phi^2 \;\;;\;\; \hat{\Omega} = \hat{\Omega}\,(\hat{V},\, \widehat{\sin\bar{\mu}c}/\bar{\mu}) \qquad \hat{\mathbb{P}} \equiv \displaystyle\int d\lambda e^{-i\lambda\hat{C}_0/\hbar} = \delta(\hat{C}_0)$

- $\mathbb{H} = Span\{\,|\pm, P_\phi)\} = Span\{\,|\Omega, \pm)\}$; $P_\phi,\, \Omega \in \mathbb{R}$

FRW Cosmology with massless ϕ

- KG theory in given $T, V_{(T)}$:

$$S_\phi^{KG}[\phi(T)] = \int dT \cdot V_{(T)} \cdot \left(\frac{d}{dT}\phi(T)\right)^2 = \int dt \; N(t)^{-1} \cdot V(t) \cdot \left(\dot{\phi}(t)\right)^2 \quad (N(t) \equiv \dot{T}(t))$$

$$= \int dt \;\; \dot{\phi}\, P_\phi - (N\, V^{-1} \cdot P_\phi^2)$$

- Coupling to Dynamical Spacetime $N(t), V(t) \equiv e^{3\,\alpha(t)}$:

$$S[\phi, P_\phi, \alpha, P_\alpha, N] = \int dt \;\; \dot{\phi}\, P_\phi + \dot{\alpha}\, P_\alpha - \bar{N}\left[P_\phi^2 - (P_\alpha^2 + e^{6\alpha})\right] \quad (\bar{N} = N V^{-1})$$

- Hamiltonian $\bar{N}\left[P_\phi^2 - (P_\alpha^2 + e^{6\alpha})\right] \equiv \bar{N}\, C \;\longrightarrow\; C = 0$

Classical Dynamics

*Reduced Phase Space

$$S[\phi, P_\phi, \alpha, P_\alpha, N]$$

$$\phi(t) = t$$

$$C = 0 \;\rightarrow\; P_\phi = P_\phi(\alpha, P_\alpha, t)$$

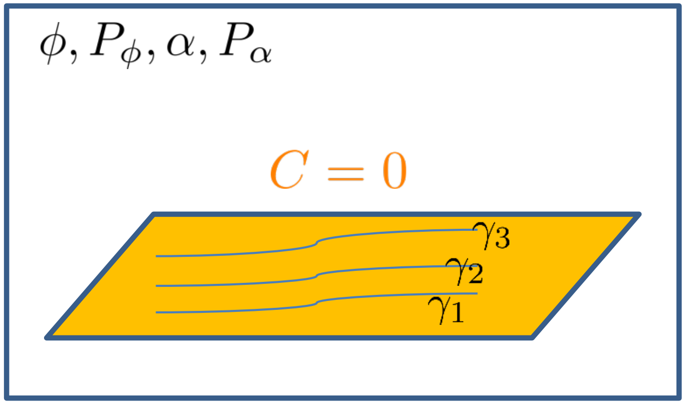
$$\phi(t), P_\phi(\alpha, P_\alpha, t) \;\rightarrow\; N(\alpha, P_\alpha, t)$$

$$S_{red}[\alpha(\phi), P_\alpha(\phi)]$$

Physical Hamiltonian $h_\phi[\alpha, P_\alpha, \phi]$

$$\{\alpha, P_\alpha\} = 1$$

*Dirac Approach



$$O(\phi, P_\phi, \alpha, P_\alpha)\Big|_{\gamma_n} = \; const$$

$$\{O_i\} \subset \{\; \alpha(\phi), P_\alpha(\phi), ...$$

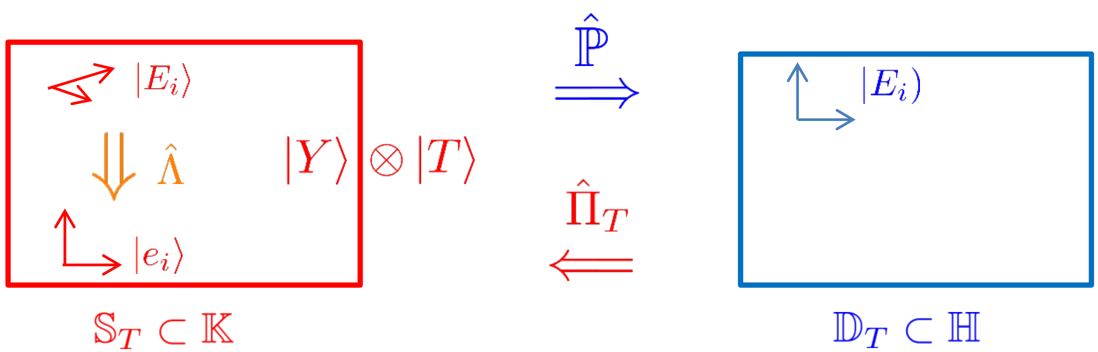
$$...., \phi(\tau_{(\alpha, P_\alpha)}), P_\phi(\tau_{(\alpha, P_\alpha)}) \;\}$$

$$\{\alpha(\phi), P_\alpha(\phi)\} = 1$$

General Construction: $\hat{C}(\hat{X}, \hat{P}_X, \hat{Y}, \hat{P}_Y)$.

- $\mathbb{K} \equiv Span\{|X, Y\rangle\}$ $\quad \begin{cases} \hat{\mathbb{P}} \equiv \int d\lambda\; e^{-i\lambda\hat{C}/\hbar} = \delta(\hat{C}) \;:\; \mathbb{K} \to \mathbb{H} \;(\, |\psi\rangle \mapsto |\Psi\rangle \,) \\ (\Psi'|\Psi) \equiv \langle\psi'|\; \hat{\mathbb{P}}\; |\psi\rangle \end{cases}$

- Choose $\hat{T} \equiv \hat{T}(\hat{X}, \hat{P}_X)$:



- Complete Obsv : $\begin{cases} \left(\hat{Y}^\Lambda(T),\; \hat{P}_Y^\Lambda(T)\right) \equiv \hat{\mathbb{P}}\, \hat{\Lambda}^{-1}\; \left(\hat{Y}, \hat{P}_Y\right)\; \hat{\Lambda}\, \hat{\Pi}_T \quad \left[\hat{Y}^\Lambda(T),\; \hat{P}_Y^\Lambda(T)\right] \text{ isomorph to} \\ \left\{ |Y^\Lambda(T)) = \hat{\mathbb{P}}\; \hat{\Lambda}\; |T, Y\rangle \right\} \text{ Orthnml Basis for } \mathbb{D}_T \end{cases}$

- Algorithm : $\hat{T} \overset{\langle \mathbb{S}'_T|\, \hat{\mathbb{P}}\, |\mathbb{S}_T\rangle}{\Longrightarrow}$ kernel ? ; $\hat{\Lambda} \overset{\langle \mathbb{S}'_T|\hat{\Lambda}\; \hat{\mathbb{P}}\; \hat{\Lambda}|\mathbb{S}_T\rangle}{\Longrightarrow} \mathbb{D}_T = \mathbb{D}$? $\Longrightarrow \Psi_{\mathbb{D}}[Y^\Lambda](T)$

FRW LQC + massless K.G Scalar

$$\hat{\alpha}, \hat{P}_\alpha, \hat{h}_\phi[\hat{\alpha}, \hat{P}_\alpha, \phi] \longrightarrow (\alpha'(\phi')|\alpha(\phi)) = U_{\alpha'(\phi'),\alpha(\phi)}$$

- Dirac quantization:

- $$\hat{\phi}, \hat{P}_\phi, \hat{\alpha}, \hat{P}_\alpha, \hat{C} \equiv \left[\hat{P}_\phi^2 - (\hat{P}_\alpha^2 + e^{6\hat{\alpha}})\right] : \mathbb{K} \to \mathbb{K} \quad \textit{self adjoint}$$

$$\Rightarrow \mathbb{K} \equiv Span\left\{\int d\lambda dP_\phi\, \psi(\lambda, P_\phi)\, |K_{i\lambda}, P_\phi\rangle\right\}_{\psi \subset L^2(\mathbb{R}^2)} \; ; \; \langle K_{i\lambda}|\alpha\rangle \equiv K_{i\lambda}(\alpha)$$

- $$\hat{\mathbb{P}} \equiv \int d\bar{N} e^{-i\bar{N}\hat{C}/\hbar} = \delta(\hat{C}) : \mathbb{K} \to \mathbb{H} \; ; \quad \hat{\mathbb{P}}\, |\alpha, \phi\rangle \equiv |\Psi_{\alpha,\phi})$$

$$\langle \alpha', \phi'|\, \hat{\mathbb{P}}\, |\alpha, \phi\rangle \equiv (\Psi_{\alpha',\phi'}|\Psi_{\alpha,\phi}) \neq U_{\alpha'(\phi'),\alpha(\phi)} \quad \textit{Non unitary!}$$

- $$\{\, \widehat{\alpha(\phi)}, \widehat{P_\alpha(\phi)}, \ldots, \phi(\widehat{\tau_{(\alpha,P_\alpha)}}), P_\phi(\widehat{\tau_{(\alpha,P_\alpha)}}) \,\} : \mathbb{H} \to \mathbb{H}$$

Path-Integral Representation

- Reduced Phase Space quantization:

$$(\alpha'(\phi')|\alpha(\phi)) = \int_{\alpha'(\phi')}^{\alpha(\phi)} D^{(\rho)}\alpha \, D^{(\rho)}P_\alpha \;\; e^{-iS^{(\rho)}_{red}[\alpha(\phi),P_\alpha(\phi)]}$$

- Dirac Quantization:

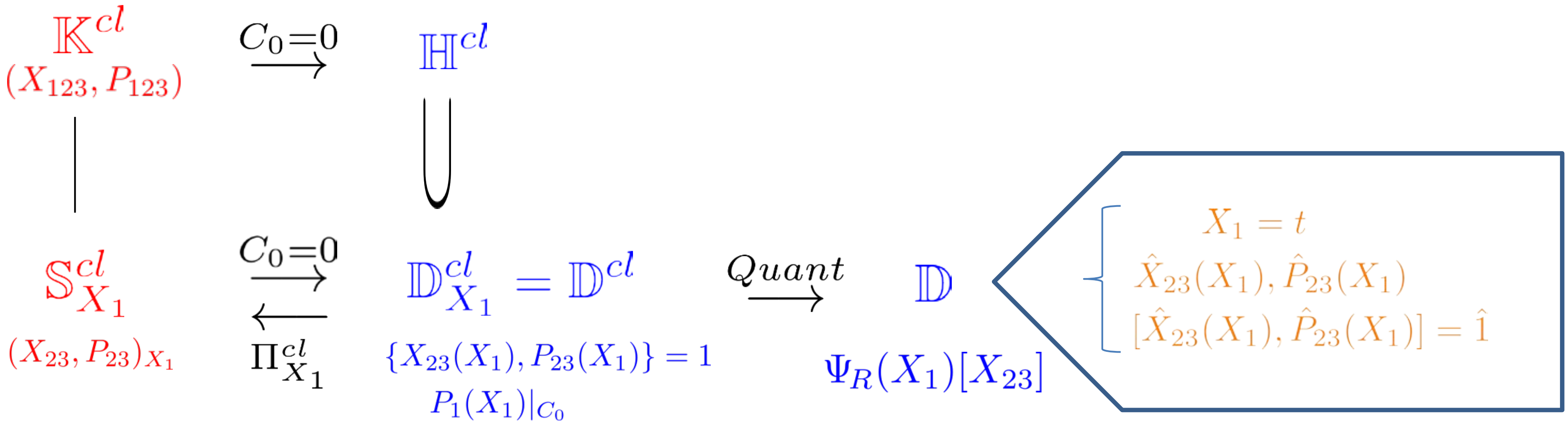
$$(\Psi_{\alpha',\phi'}|\Psi_{\alpha,\phi}) = \int_{(\alpha',\phi')(t_1)}^{(\alpha,\phi)(t_0)} D^{(\eta)}\alpha \, D^{(\eta)}P_\alpha \, D^{(\eta)}\phi \, D^{(\eta)}P_\phi \, D^{(\eta)}N$$

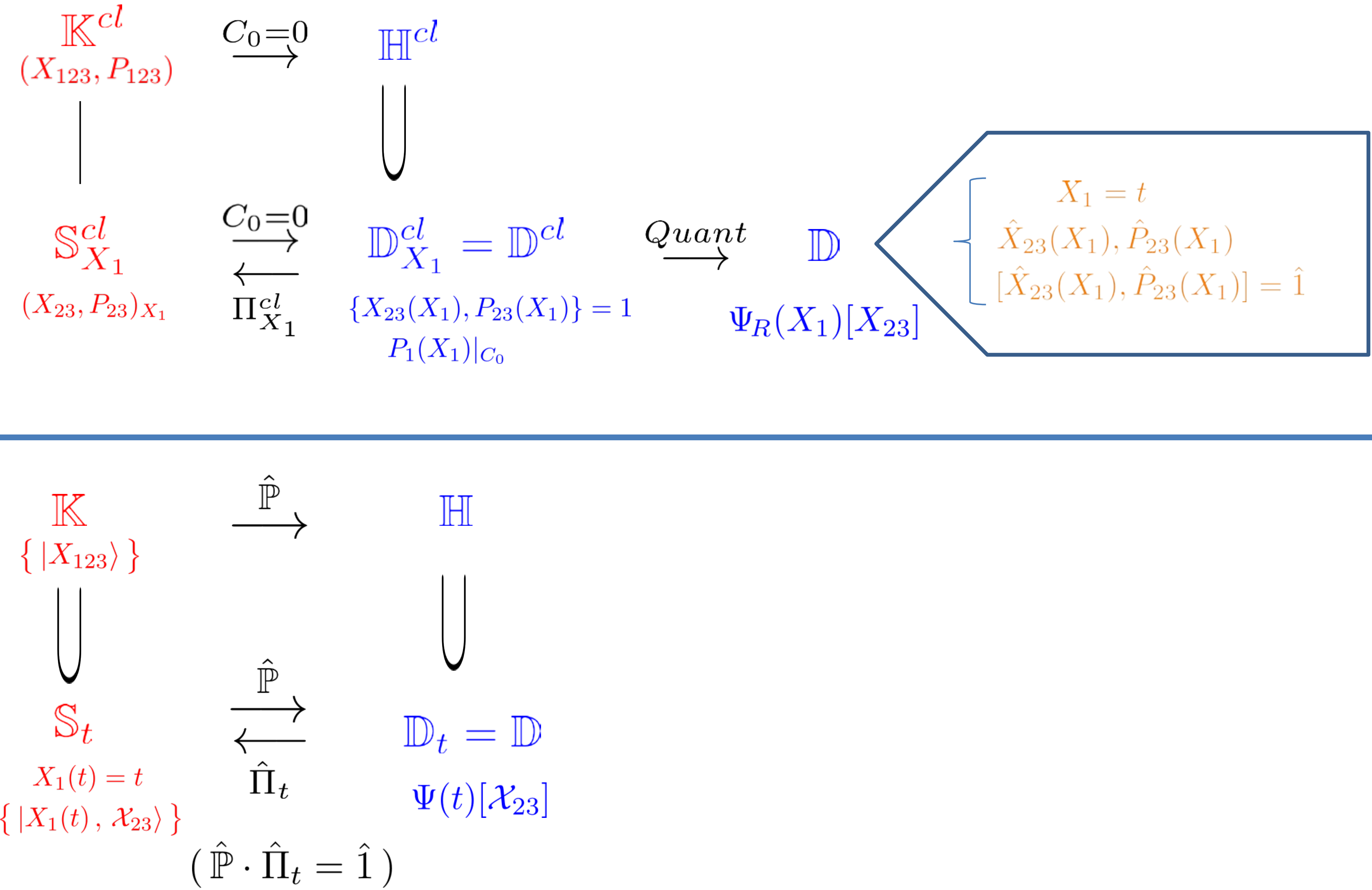
$$\times \, |P_\phi| \, \delta(\phi(t)-t) \, \cdot \, e^{-iS^{(\eta)}[\alpha(t),P_\alpha(t),\phi(t),P_\phi(t),N(t)]}$$

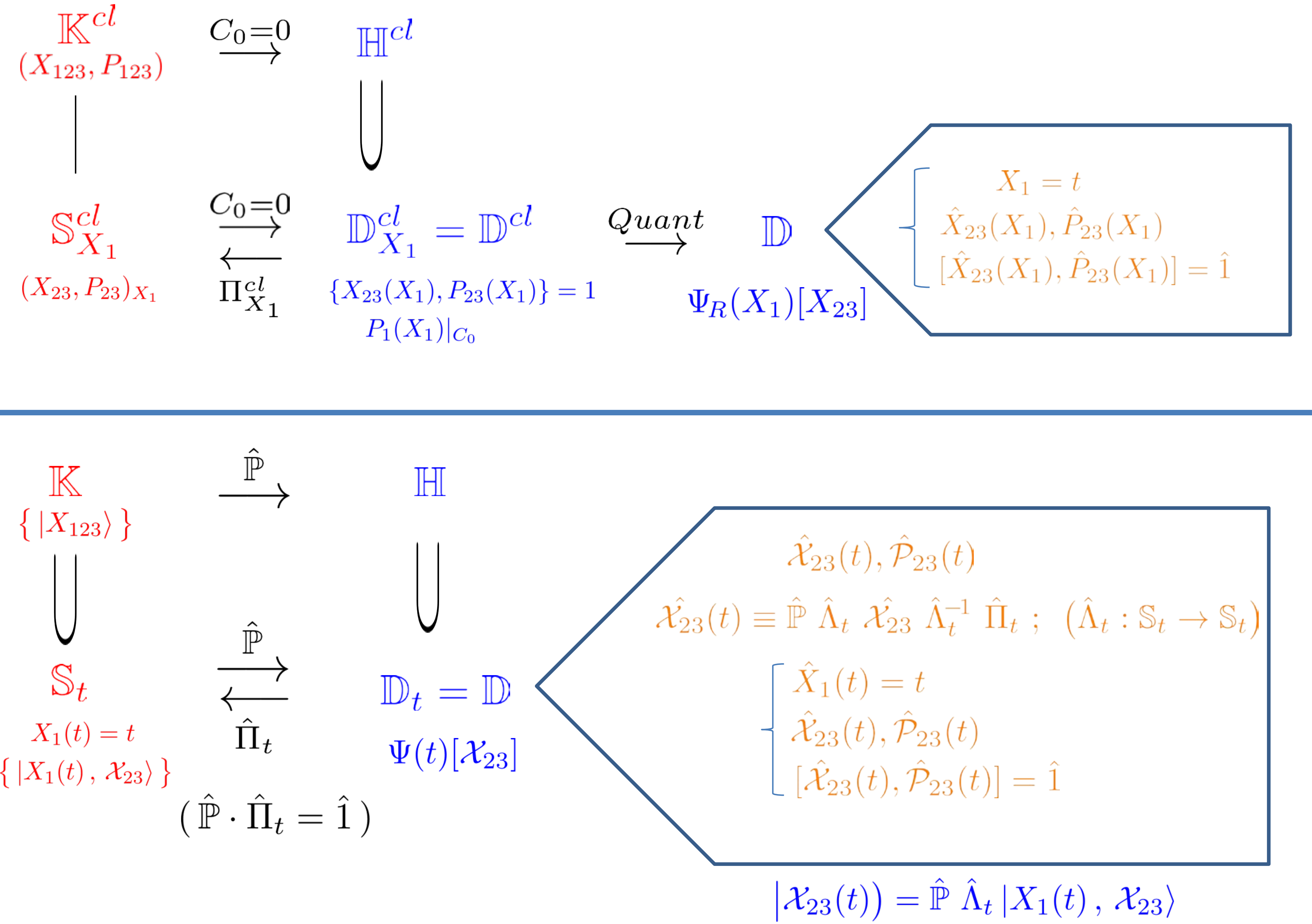
- Barvinsky et al : $(\alpha'(\phi')|\alpha(\phi)) = \Lambda(t_1)\cdot\Lambda(t_0)\cdot(\Psi_{\alpha',\phi'}|\Psi_{\alpha,\phi})\big|_{\phi(t)=t} + O(\hbar)$

$$\Lambda(t) = |\{\phi, C\}(t)|^{1/2} = \sqrt{|P_\phi(t)|}$$

$$
\begin{array}{ccc}
\mathbb{K}^{cl} & \xrightarrow{C_0=0} & \mathbb{H}^{cl} \\
(X_{123}, P_{123}) & & \\
\Big| & & \bigcup \\
\mathbb{S}^{cl}_{X_1} & \underset{\Pi^{cl}_{X_1}}{\overset{C_0=0}{\rightleftarrows}} & \mathbb{D}^{cl}_{X_1} = \mathbb{D}^{cl} \\
(X_{23}, P_{23})_{X_1} & & \{X_{23}(X_1), P_{23}(X_1)\} = 1 \\
 & & P_1(X_1)|_{C_0}
\end{array}
$$


$\mathbb{K}^{cl}$
(X_{123}, P_{123})
$C_0=0$
$\mathbb{H}^{cl}$
$\mathbb{S}^{cl}_{X_1}$
$(X_{23}, P_{23})_{X_1}$
$C_0=0$
$\Pi^{cl}_{X_1}$
$\mathbb{D}^{cl}_{X_1} = \mathbb{D}^{cl}$
$\{X_{23}(X_1), P_{23}(X_1)\} = 1$
$P_1(X_1)|_{C_0}$
$Quant$
$\mathbb{D}$
$\Psi_R(X_1)[X_{23}]$
$X_1 = t$
$\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)$
$[\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)] = \hat{1}$


$\mathbb{K}^{cl}$
(X_{123}, P_{123})
$\xrightarrow{C_0=0}$
$\mathbb{H}^{cl}$
$\mathbb{S}^{cl}_{X_1}$
$(X_{23}, P_{23})_{X_1}$
$\xrightarrow{C_0=0}$
$\Pi^{cl}_{X_1}$
$\mathbb{D}^{cl}_{X_1} = \mathbb{D}^{cl}$
$\{X_{23}(X_1), P_{23}(X_1)\} = 1$
$P_1(X_1)|_{C_0}$
$\xrightarrow{Quant}$
$\mathbb{D}$
$\Psi_R(X_1)[X_{23}]$
$X_1 = t$
$\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)$
$[\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)] = \hat{1}$
$\mathbb{K}$
$\{ |X_{123}\rangle \}$
$\xrightarrow{\hat{\mathbb{P}}}$
$\mathbb{H}$
$\mathbb{S}_t$
$X_1(t) = t$
$\{ |X_1(t), \mathcal{X}_{23}\rangle \}$
$\xrightarrow{\hat{\mathbb{P}}}$
$\hat{\Pi}_t$
$\mathbb{D}_t = \mathbb{D}$
$\Psi(t)[\mathcal{X}_{23}]$
$(\hat{\mathbb{P}} \cdot \hat{\Pi}_t = \hat{1})$



$\mathbb{K}^{cl}$
(X_{123}, P_{123})
$\xrightarrow{C_0=0}$
$\mathbb{H}^{cl}$
$\mathbb{S}^{cl}_{X_1}$
$(X_{23}, P_{23})_{X_1}$
$\xrightarrow{C_0=0}$
$\Pi^{cl}_{X_1}$
$\mathbb{D}^{cl}_{X_1} = \mathbb{D}^{cl}$
$\{X_{23}(X_1), P_{23}(X_1)\} = 1$
$P_1(X_1)|_{C_0}$
$\xrightarrow{Quant}$
$\mathbb{D}$
$\Psi_R(X_1)[X_{23}]$
$X_1 = t$
$\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)$
$[\hat{X}_{23}(X_1), \hat{P}_{23}(X_1)] = \hat{1}$
$\mathbb{K}$
$\{\, |X_{123}\rangle \,\}$
$\xrightarrow{\hat{\mathbb{P}}}$
$\mathbb{H}$
$\mathbb{S}_t$
$X_1(t) = t$
$\{\, |X_1(t)\,,\, \mathcal{X}_{23}\rangle \,\}$
$\xrightarrow{\hat{\mathbb{P}}}$
$\hat{\Pi}_t$
$(\, \hat{\mathbb{P}} \cdot \hat{\Pi}_t = \hat{1} \,)$
$\mathbb{D}_t = \mathbb{D}$
$\Psi(t)[\mathcal{X}_{23}]$
$\hat{\mathcal{X}}_{23}(t), \hat{\mathcal{P}}_{23}(t)$
$\hat{\mathcal{X}}_{23}(t) \equiv \hat{\mathbb{P}}\ \hat{\Lambda}_t\ \hat{\mathcal{X}}_{23}\ \hat{\Lambda}_t^{-1}\ \hat{\Pi}_t\ ;\ \ (\hat{\Lambda}_t : \mathbb{S}_t \to \mathbb{S}_t)$
$\hat{X}_1(t) = t$
$\hat{\mathcal{X}}_{23}(t), \hat{\mathcal{P}}_{23}(t)$
$[\hat{\mathcal{X}}_{23}(t), \hat{\mathcal{P}}_{23}(t)] = \hat{1}$
$\big|\mathcal{X}_{23}(t)\big) = \hat{\mathbb{P}}\ \hat{\Lambda}_t\, |X_1(t)\,,\, \mathcal{X}_{23}\rangle$