About the Kohn-Dirac operator on CR manifolds

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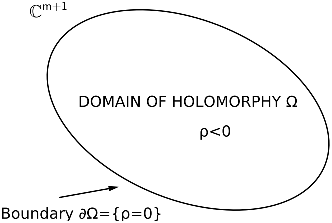
Strictly pseudo-convex boundaries



Let $\Omega = \{\rho < 0\} \subseteq \mathbb{C}^{m+1}$ $(m \geq 1)$ be domain of holomorphy with defining C^∞-function ρ

$\Rightarrow$ **Levi form** is positive semidefinite

$$\mathcal{L}(w, w) = \sum_{j,k=0}^{m} \frac{\partial \rho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0, \qquad w \in T^{10}(\partial\Omega)$$

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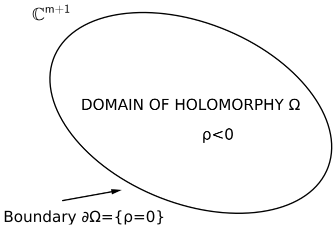

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If $\mathcal{L}(w, w) > 0$ for $w \neq 0$ then

- $H = T(\partial\Omega) \cap J(T(\partial\Omega))$ is contact on $\partial\Omega$
- (H, J) is **CR structure** on $M := \partial\Omega$
- with contact 1-form $\theta \ = \ d\rho \circ J$

We call directions in $\mathbb{C} \otimes H \ = \ H^{10} \oplus H^{01}$ **transverse**

Abstract CR manifolds with pseudo-Hermitian form

Let (M^{2m+1}, H, J) be strictly Ψ-convex CR manifold with adapted Ψ-Hermitian 1-form θ (i.e. $H = ker\theta$)

$\hookrightarrow$ characteristic vector $T = T^{\theta}$ on M

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and **Tanaka-Webster connection** ∇^θ on H

- metric connection w.r.t. $g_\theta = \frac{1}{2}d\theta(\cdot, J\cdot)$ on H
- $Tor^\nabla(X, Y) = d\theta(X, Y)T$ for $X, Y \in H$
- Webster torsion $\tau(X) = -\frac{1}{2}([T, X] + J[T, JX])$
- $U(m)$ is structure group of ∇^θ

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$\hookrightarrow$

$$\rho_\theta^{ric} = \frac{scal_\theta}{4m} \cdot d\theta \qquad \textbf{Ψ-Einstein condition}$$

CR spinors and Kohn-Dirac operator

If $c_1(\mathcal{K}) \equiv 0 \; mod \; 2$, the transverse distribution

$(H, g_\theta) \to M$ admits **spinor bundle** $\Sigma(M)$ with

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- inner product $(\cdot, \cdot)_{L^2}$ on sections $\Gamma_o(\Sigma)$
- Tanaka-Webster spinor connection ∇^Σ
- $\Sigma(M) = \bigoplus_{r=0}^{m} \Sigma_{\mu_r}$ ($\mu_r = m - 2r$ eigenvalues of $\frac{id\theta}{2}$)
- **Kohn-Dirac operator**

$$D_\theta \Phi = \sum_{j=1}^{2m} e_i \cdot \nabla^\Sigma_{e_i} \Phi, \qquad \Phi \in \Gamma(\Sigma)$$

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Note: Eigenspinors $D_\theta \Phi = \lambda \cdot \Phi$ with $\lambda \neq 0$ decompose to

$$\Phi = \Phi_{\mu_r} + \Phi_{\mu_{r+1}} \in \Gamma(\Sigma_{\mu_r} \oplus \Sigma_{\mu_{r+1}}) \qquad \text{of type } (r, r+1)$$

Analytic properties of D_θ

Let (M^{2m+1}, H, J, θ) be **closed** spin CR manifold $(m \geq 2)$

- D_θ is formally self-adjoint
- D_θ is **not** elliptic
- the square

  $$D_\theta^2 : \Gamma(\Sigma_{\mu_r}) \to \Gamma(\Sigma_{\mu_r})$$

  is hypoelliptic for the non-extremal bundles $r \neq 0, m$
- $spec(D_\theta^2)$ is pure point spectrum ν_i
  with $\nu_i \geq 0$ and $\nu_i \to \infty$ (Stadtmüller '17)
- eigenspaces E_{ν_i} are finite-dimensional for $\nu_i > 0$
- in general, the space of harmonic spinors has $dim\mathcal{H} = \infty$

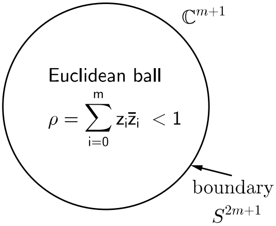
Example: The standard CR sphere

The Euclidean ball $B_1(0) \subseteq \mathbb{C}^{m+1}$ is convex

boundary = sphere S^{2m+1} of radius 1,

Ψ-Herm.form $\quad \theta_o = d\rho \circ J$,

$scal_{\theta_o} \equiv 4m(m+1) = const.$



as **homogeneous space:**

$S^{2m+1} = \tilde{U}(m+1)/\tilde{U}(m)$ with $\tilde{U}(m+1) \subseteq Spin(2m+2)$

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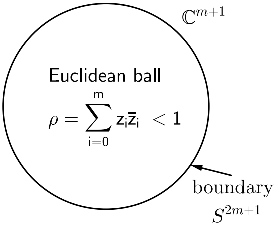

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By **Frobenius reciprocity** we have

$$L^2(\Sigma) \ \cong \ \overline{\bigoplus_{\gamma} V_\gamma \otimes Hom_{\tilde{U}(m)}(V_\gamma, \Sigma)}$$

where $\qquad V_\gamma = (V_{\gamma(a,b)}, \pi) \qquad (a, b \geq 0)$

are the (relevant) irreducible representations of $\tilde{U}(m+1)$

Spectrum of S^{2m+1}

Parthasarathy-type formula

$$D^2_{\theta_o}(v \otimes A) \;=\; v \otimes \left\{ \begin{array}{c} A \circ \pi_*(casimir_{\mathfrak{u}(m+1)}) \;+\; \frac{scal_{\theta_o}}{8} \cdot A \\ +\, 2 \cdot A \circ \pi_*(T)^2 + d\theta_o \cdot A \circ \pi_*(T) + \frac{1}{2} d\theta_o^2 \cdot A \end{array} \right\}$$

for $A \in Hom_{\tilde{U}(m)}(V_{\gamma(a,b)}, \Sigma)$ with $a, b \geq 0$

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This gives the **eigenvalues**

$$\{\, \lambda = \pm\sqrt{4(m-r+a)(r+1+b)} \mid \; a, b \geq 0 \;\; \wedge \;\; 0 \leq r < m\}$$

with **lower bound** for the non-zero eigenvalues of type $(r, r+1)$:

$$\lambda^2 \;\; \geq \;\; 4(m-r)(r+1) \cdot \frac{scal_{\theta_o}}{4m(m+1)}$$

Schrödinger-Lichnerowicz-type formula

$$
\begin{aligned}
D_\theta &= \quad \text{Kohn-Dirac operator} \\
\Delta^\Sigma &= -\, tr_H(\nabla^\Sigma \circ \nabla^\Sigma) \qquad \text{spinor sub-Laplacian} \\
&= \quad \Delta_{10} + \Delta_{01} \qquad\qquad (\text{ w.r.t. } \quad J \text{ on } H\ ) \\
\nabla^\Sigma_T &= -\frac{1}{4m}\rho_\theta^{ric} \cdot \; + \frac{i}{2m}\left(\Delta_{10} - \Delta_{01}\right)
\end{aligned}
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$$\nabla^\Sigma_T = -\frac{1}{4m}\rho^{ric}_\theta \cdot \; + \frac{i}{2m}(\Delta_{10} - \Delta_{01})$$

Lichnerowicz-type formula (Petit '05):

$$D^2_\theta = \Delta^\Sigma + \frac{scal_\theta}{4} - d\theta \cdot \nabla^\Sigma_T$$

$$\Rightarrow \quad D^2_\theta = \left(1 - \frac{id\theta}{2m}\right)\Delta_{10} + \left(1 + \frac{id\theta}{2m}\right)\Delta_{01}$$
$$+\frac{1}{4}\left(scal_\theta + \frac{1}{m} d\theta \cdot \rho^{ric}_\theta \right)$$

Lower bounds for eigenvalues

Let (M^{2m+1}, θ), $m \geq 2$, be closed with positive Ricci-form $\rho_\theta^{ric} > 0$

Set

$$s_o := \frac{scal_{min}}{4m(m+1)}$$

If $\Phi = \Phi_{\mu_r} + \Phi_{\mu_{r+1}}$ is some $(r, r+1)$-eigenspinor for $\lambda \neq 0$ then

$$\lambda^2 \quad \geq \quad s_o \cdot \begin{cases} \frac{2m^2}{m-1} & \text{for} \quad r = 0, m-1 \\ m(m+2) & \text{for} \quad r = \frac{m}{2}, \frac{m-2}{2} \\ m(m+1) & \text{for} \quad r = \frac{m-1}{2} \\ \frac{2(r+1)(m-r)m}{m-r+1} & \text{for} \quad 0 < r < \frac{m-2}{2} \\ \frac{2(r+1)(m-r)m}{r} & \text{for} \quad \frac{m}{2} < r < m-1 \end{cases}$$

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On Ψ-**Einstein spaces** (M^{2m+1},θ) the lower bound is given by

$$\lambda^2 \quad \geq \quad 4(r+1)(m-r)s_o$$

Ψ-Killing spinors

If $\Phi = \Phi_{\mu_r} + \Phi_{\mu_{r+1}}$ realizes the general lower bound for λ^2

$\Rightarrow$ Φ_{μ_r} or $\Phi_{\mu_{r+1}}$ satisfiy some **twistor equation**

in the simultaneous case:

$$\nabla^{\Sigma}_{X_{10}} \Phi_{\mu_r} = 0, \qquad \nabla^{\Sigma}_{X_{01}} \Phi_{\mu_{r+1}} = 0$$

$$\nabla^{\Sigma}_{X_{01}} \Phi_{\mu_r} = -\tfrac{\lambda}{2(r+1)} X_{01} \Phi_{\mu_{r+1}}$$

$$\nabla^{\Sigma}_{X_{10}} \Phi_{\mu_{r+1}} = -\tfrac{\lambda}{2(m+r)} X_{10} \Phi_{\mu_r}$$

for any **transversal vector** $X = X_{10} + X_{01} \in H$

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for any **transversal vector** $X = X_{10} + X_{01} \in H$

$\hookrightarrow$ $\Phi =$ strong (transversal) **Ψ-Killing spinors**

Example: Such Φ exist on 3-**Sasakian spaces**

(with related Ψ-Hermitian form θ)

Parallel spinors

We call $\Phi \in \Gamma(\Sigma)$ **transversally parallel** iff

$$\nabla^{\Sigma}_{X}\Phi \;\; = \;\; 0 \qquad \forall X \in H$$

Due to torsion $R(X, JY)\Phi = d\theta(X, JY)\nabla^{\Sigma}_{T}\Phi$ and

$$\nabla^{\Sigma}_{T}\Phi \;\; = \;\; -\frac{1}{2m}\rho^{ric}_{\theta} \cdot \Phi$$

If we **modify** the Tanaka-Webster connection by

$$\hat{\nabla}_{T}\Phi \; := \; \nabla^{\Sigma}_{T}\Phi \; + \; \frac{1}{2m}\rho^{ric}_{\theta} \cdot \Phi$$

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Computing the curvature $\hat{R}$ shows:

θ is Ψ-Einstein iff $tr_{\mathbb{C}}\hat{R}(X, Y) = 0 \quad \forall X, Y \in TM$

i.e. the **holonomy algebra** is reduced: $\mathfrak{hol}(\hat{\nabla}) \;\subseteq\; \mathfrak{su}(m)$

Characterization of Ψ-Einstein spaces

Theorem: Let (M^{2m+1}, θ) be 1-connected. Then we have the equivalent conditions:

- M admits transversally parallel spinors
- **basic holonomy** $Hol(\theta) \subseteq SU(m)$
- θ is Ψ-Einstein

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Existence:

(1) For $M = \partial\Omega \subseteq \mathbb{C}^{m+1}$ strictly Ψ-convex boundary

- $vol_{\mathbb{C}} = dz_0 \wedge \cdots \wedge dz_m$ induces Ψ-Einstein structure on $\partial\Omega$
- $H(M)$ is spin with transversally parallel spinor

(2) S^1-bundles L with $c_1(L) = [\omega] \in H^2(M, \mathbb{Z})$
over compact hyperKähler manifolds have $Hol(\theta) = Sp(\frac{m}{2})$

CR invariants

Let $\tilde{\theta} = e^{2\sigma}\theta$ be **another** adapted Ψ-Hermitian form on (M, H, J)

Then

$$D_0 := D_\theta|_{\Gamma(\Sigma_0)} \quad : \; \Gamma(\Sigma_0) \to \Gamma(\Sigma_2 \oplus \Sigma_{-2})$$

is **CR-covariant** for $m =$ even, i.e.

$$D_{\tilde{\theta}}(e^{-(m+1)\sigma} \cdot \tilde{\Phi}_0) \quad = \quad e^{(-m+2)\sigma} \cdot \widetilde{D_\theta \Phi_0}$$

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In particular,

$$\mathcal{H}_0 \quad = \quad \text{space of harmonic spinors in } \Gamma(\Sigma_0)$$

is a **CR invariant**

Vanishing Theorems

Lichnerowicz-type formula for spinors in $\Gamma(\Sigma_0)$:

$$D_0^* D_0 \;=\; \Delta^{\Sigma} \;+\; \frac{scal_\theta}{4}$$

$\Rightarrow$

- **vanishing theorem:** If $scal_\theta > 0$ on closed (M, θ) then $\mathcal{H}_0 = \{0\}$ (no harmonic spinors)
- vanishing of twisted **Kohn-Rossi** cohomology group:

$$H^{\frac{m}{2}}(M, \sqrt{\mathcal{K}}) \;\cong\; \mathcal{H}_0 \;=\; \{0\}$$

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- **obstruction:** If Kohn-Rossi group $H^{\frac{m}{2}}(M, \sqrt{\mathcal{K}}) \neq \{0\}$, then **no** θ with positive $scal_\theta > 0$ exists on M

Examples for obstruction:

(1) compact quotients of Heisenberg groups

(2) S^1-bundle with basic holonomy $Hol(\theta) = Sp(\frac{m}{2})$

$Spin^{\mathbb{C}}$-structures

Let (M^{2m+1}, H, J) be given some $Spin^{\mathbb{C}}$-structure with determinant bundle

$$L_{det} = \mathcal{E}^{\ell} \ , \qquad \ell \in \mathbb{Z},$$

where $\mathcal{E} = \sqrt[m+2]{\mathcal{K}^{-1}}$

$\hookrightarrow$ D_ℓ Kohn-Dirac operator of **weight** ℓ

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Then

- The μ_r-component of D_ℓ with $\mu_r = -\ell$ is CR-covariant
- obtain more vanishing theorems for Kohn-Rossi groups
- further obstructions for

$$\rho_\theta^{ric} > 0 \qquad \text{and} \qquad scal_\theta > 0$$

  in terms of Kohn-Rossi groups

More vanishing theorems

Lichnerowicz-type formula for D_ℓ of weight $\ell \in \mathbb{Z}$:

$$\begin{aligned} D_\ell^2 \quad = \quad & \left(1 - \frac{id\theta}{2m}\right) \Delta_{10} \quad + \quad \left(1 + \frac{id\theta}{2m}\right) \Delta_{01} \\ & - \frac{i}{2} \left(\frac{\ell}{m+2} + \frac{id\theta}{2m} \right) \rho_\theta^{ric} \quad + \quad \left(1 + \frac{i\ell d\theta}{2m(m+2)} \right) \cdot \frac{scal_\theta}{4} \end{aligned}$$

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Lichnerowicz-type formula for D_ℓ of weight $\ell \in \mathbb{Z}$:

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Theorem. Let M^{2m+1} be closed with $m \geq 2$.

If $|\ell| \leq m+2$ and $\rho_\theta > 0$, then any harmonic spinor is in the extremal bundles $\Sigma_m \oplus \Sigma_{-m}$

In particular, the qth Kohn-Rossi group $H^{0,q}(M)$ vanishes for $q = 1, \ldots, m-1$.