Weighing the spacetime along the line of sight

Michele Grasso
● Mikołaj Korzyński
Julius Serbenta
Eleonora Villa

Centre for Theoretical Physics, Polish Academy of Sciences
Warsaw

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• Determining mass distribution in the spacetime = one of the fundamental problems of astronomy
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Most methods known use gravity
Idea

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• Two possibilities
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Two possibilities

1. Use the motions of test bodies in the gravitational field

2. Use the impact of gravity on light propagation - gravitational light bending, lensing, Shapiro delays…
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Idea: new, very direct method of measuring the amount of matter along the line of sight using 2.
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Idea: new, very direct method of measuring the amount of matter along the line of sight using 2.

Based on simultaneous measurement of the parallax and: either the apparent size of an object (standard ruler) or the apparent luminosity of that object (standard candle)
Idea

- Determining mass distribution in the spacetime = one of the fundamental problems of astronomy
- Most methods known use gravity
- Two possibilities
  1. Use the motions of test bodies in the gravitational field
  2. Use the impact of gravity on light propagation - gravitational light bending, lensing, Shapiro delays…
- **Idea:** new, very direct method of measuring the amount of matter along the line of sight using 2.
- Based on simultaneous measurement of the parallax and: either the apparent size of an object (standard ruler) or the apparent luminosity of that object (standard candle)
- **Possible application:** dark and ordinary matter mapping, cosmological isotropy tests, …
Hand-waving explanation
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- Flat spacetime, no matter: measurements of distance to an object by parallax ($D_{\text{par}}$) and by angular size ($D_{\text{ang}}$) must give the same result.
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\[ D_{\text{par}} = \frac{b}{\alpha} \]
Flat spacetime, no matter: measurements of distance to an object by parallax ($D_{\text{par}}$) and by angular size ($D_{\text{ang}}$) must give the same result.

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Hand-waving explanation

- Flat spacetime, no matter: measurements of distance to an object by parallax ($D_{\text{par}}$) and by angular size ($D_{\text{ang}}$) must give the same result:

  \[ D_{\text{ang}} = \frac{b}{\alpha} \]
  \[ D_{\text{par}} = \frac{b}{\alpha} \]
  \[ D_{\text{par}} = D_{\text{ang}} \]

- Matter present along the line of sight $\Rightarrow$ gravitational light bending. Both distance measurements affected, but affected *differently!*
Hand-waving explanation

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\[D_{\text{par}} = D_{\text{ang}}\]

- Matter present along the line of sight $\Rightarrow$ gravitational light bending. Both distance measurements affected, but affected differently!

\[
D_{\text{ang}} = \frac{b}{\alpha'}
\]
\[
D_{\text{par}} = \frac{b}{\alpha'}
\]
\[\alpha' < \alpha \Rightarrow D_{\text{par}} > D_{\text{ang}}\]
• Flat spacetime, no matter: measurements of distance to an object by parallax ($D_{\text{par}}$) and by angular size ($D_{\text{ang}}$) must give the same result.

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• Objects appear further away when the distance is measured by parallax than by the angular size!
• Flat spacetime, no matter: measurements of distance to an object by parallax ($D_{\text{par}}$) and by angular size ($D_{\text{ang}}$) must give the same result

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• Objects appear further away when the distance is measured by parallax than by the angular size!

• **Claim:** the difference measures the amount of matter along the LOS between $S$ and $O$
Hand-waving explanation

\[ D_{\text{ang}} = \frac{b}{\alpha} \]

\[ D_{\text{par}} = \frac{b}{\alpha'} \]

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Hand-waving explanation

- What about light bending by matter off the line of sight?

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\( \alpha' < \alpha \implies D_{\text{par}} > D_{\text{ang}} \)
Hand-waving explanation

- What about light bending by matter off the line of sight?

- What about the motions of the source and the observer and the special relativistic effects (time dilation, stellar aberration...)?

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D_{\text{par}} = \frac{b}{\alpha'} \\
\alpha' < \alpha \implies D_{\text{par}} > D_{\text{ang}}
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Hand-waving explanation

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- What about the motions of the source and the observer and the special relativistic effects (time dilation, stellar aberration...)?

- Directional dependence of the effect: need to consider 2D angles and displacements

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• What about light bending by matter off the line of sight?

• What about the motions of the source and the observer and the special relativistic effects (time dilation, stellar aberration…)?

• Directional dependence of the effect: need to consider 2D angles and displacements

• Need of a fully relativistic approach! (all GR and SR effects)
Fully relativistic approach
Fully relativistic approach

- Spacetime with *any* Lorentzian metric


- Spacetime with any Lorentzian metric

- Two distant, small regions of spacetime, connected by a null geodesic
Fully relativistic approach

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- Observers measure the TOA, position on the sky, drifts …

Fully relativistic approach

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- Two types of effects:

Fully relativistic approach

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- Two types of effects:
  - GR light propagation effects: 1st order GDE (linearisation in the transverse coordinates)

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- Two types of effects:
  - GR light propagation effects: 1st order GDE (linearisation in the transverse coordinates)
  - SR effects (aberration, time dilation, Doppler…) given an emitter and observer
Fully relativistic approach

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Separate the dependence in the expressions

---

Light propagation effects

\[ l + \Delta l_{\Theta} \]

\[ l + \Delta l_{\varepsilon} \]

\[ \delta x_{\Theta} \]

\[ \delta x_{\varepsilon} \]

\[ \gamma_0 \]
Light propagation effects

\[ \delta x_\mathcal{O}^\mu \]

\[ \Delta l_\mathcal{O}^\mu = \delta l_\mathcal{O}^\mu + \Gamma_\mathcal{O}^{\mu\nu\sigma} \delta x_\mathcal{O}^\nu \delta x_\mathcal{O}^\sigma \]
Light propagation effects

\[ \delta x^\mu_\mathcal{O} \]

\[ \Delta l^\mu_\mathcal{O} = \delta l^\mu_\mathcal{O} + \Gamma^\mu_{\nu\sigma}(\mathcal{O}) l^\nu \delta x^\sigma_\mathcal{O} \]

\[ l + \Delta l_\mathcal{O} \]

\[ \nabla_l \nabla_l \xi^\mu - R^\mu_{\alpha\beta\nu} l^\alpha l^\beta \xi^\nu = 0 \]

\[ \xi^\mu(\lambda_\mathcal{O}) = \delta x^\mu_\mathcal{O} \]

\[ \nabla_l \xi^\mu(\lambda_\mathcal{O}) = \Delta l^\mu_\mathcal{O} \]

\[ l + \Delta l_\mathcal{E} \]

\[ \delta x_\mathcal{E} \]

\[ \nabla_l (\lambda_\mathcal{E}) = 0 \]

\[ \nabla_l \xi^\mu(\lambda_\mathcal{E}) = \Delta l^\mu_\mathcal{E} \]
Light propagation effects

\[ \delta x_\mathcal{O}^\mu \]

\[ \Delta l_\mathcal{O}^\mu = \delta l_\mathcal{O}^\mu + \Gamma_{\nu\sigma}^\mu(\mathcal{O}) l^\nu \delta x_\mathcal{O}^\sigma \]

\[ \nabla l \nabla_\xi^\mu - R^\mu_{\alpha\beta\nu} l^\alpha l^\beta \xi^\nu = 0 \]

\[ \xi^\mu(\lambda_\mathcal{O}) = \delta x_\mathcal{O}^\mu \]

\[ \nabla_\xi^\mu(\lambda_\mathcal{O}) = \Delta l_\mathcal{O}^\mu \]

\[ \delta x_\xi^\mu = \xi^\mu(\lambda_\xi) \]

\[ \Delta l_\xi^\mu = \nabla_\xi^\mu(\lambda_\xi) \]
Light propagation effects

\[ \delta x^\mu_O \]
\[ \Delta l^\mu_O = \delta l^\mu_O + \Gamma^\mu_{\nu\sigma}(O) l^\nu \delta x^\sigma_O \]

\[ l + \Delta l_O \]

\[ \nabla_l \nabla \xi^\mu - R^\mu_{\alpha\beta\nu} l^\alpha l^\beta \xi^\nu = 0 \]
\[ \xi^\mu(\lambda_O) = \delta x^\mu_O \]
\[ \nabla_l \xi^\mu(\lambda_O) = \Delta l^\mu_O \]

\[ \delta x^\mu_\xi = \xi^\mu(\lambda_\xi) \]
\[ \Delta l^\mu_\xi = \nabla_l \xi^\mu(\lambda_\xi) \]

\[ \delta x^\mu_\xi = W_{xx}^\mu_\nu \delta x^\nu_O + W_{xl}^\mu_\nu \Delta l^\nu_O \]
\[ \Delta l^\mu_\xi = W_{lx}^\mu_\nu \delta x^\nu_O + W_{ll}^\mu_\nu \Delta l^\nu_O \]
Light propagation effects

\[ \delta x^\mu_\Theta \]
\[ \Delta l^\mu_\Theta = \delta l^\mu_\Theta + \Gamma^\mu_{\nu\sigma}(\Theta) l^\nu \delta x^\sigma_\Theta \]

\[ \nabla_l \nabla_l \xi^\mu - R^\mu_{\alpha\beta\nu} l^\alpha l^\beta \xi^\nu = 0 \]
\[ \xi^\mu(\lambda_\Theta) = \delta x^\mu_\Theta \]
\[ \nabla_l \xi^\mu(\lambda_\Theta) = \Delta l^\mu_\Theta \]

\[ \delta x^\mu_\xi = \xi^\mu(\lambda_\xi) \]
\[ \Delta l^\mu_\xi = \nabla_l \xi^\mu(\lambda_\xi) \]

\[ \delta x^\mu_\xi = W_{xx}^\mu_\nu \delta x^\nu_\Theta + W_{xl}^\mu_\nu \Delta l^\nu_\Theta \]
\[ \Delta l^\mu_\xi = W_{lx}^\mu_\nu \delta x^\nu_\Theta + W_{ll}^\mu_\nu \Delta l^\nu_\Theta \]

\[ W_{**} : T_\Theta M \mapsto T_\xi M \]
Light propagation effects

$$\delta x^\mu_O$$

$$\Delta l^\mu_O = \delta l^\mu_O + \Gamma^\mu_{\nu\sigma}(O) l^\nu \delta x^\sigma_O$$

$$l + \Delta l_O$$

$$\ddot{A}^\mu_\nu - R^\mu_{\alpha\beta\sigma} l^\alpha l^\beta A^\sigma_\nu = 0$$

$$A^\mu_\nu(\lambda_O) = \delta^\mu_\nu$$

$$\dot{A}^\mu_\nu(\lambda_O) = 0$$

$$W_{xx}^\mu_\nu = A^\mu_\nu(\lambda_\varepsilon)$$

$$W_{lx}^\mu_\nu = \dot{A}^\mu_\nu(\lambda_\varepsilon)$$

$$\delta x^\mu_\varepsilon = W_{xx}^\mu_\nu \delta x^\nu_O + W_{xl}^\mu_\nu \Delta l^\nu_O$$

$$\Delta l^\mu_\varepsilon = W_{lx}^\mu_\nu \delta x^\nu_O + W_{ll}^\mu_\nu \Delta l^\nu_O$$

$$W^{**} : T_O M \mapsto T_\varepsilon M$$
Light propagation effects

\[ \delta x^\mu_\mathcal{O} \]
\[ \Delta l^\mu_\mathcal{O} = \delta l^\mu_\mathcal{O} + \Gamma^\mu_{\nu\sigma}(\mathcal{O}) l^\nu \delta x^\sigma_\mathcal{O} \]

\[ l + \Delta l_\mathcal{O} \]

\[ \ddot{B}^\mu_\nu - R^\mu_{\alpha\beta\sigma} l^\alpha l^\beta B^\sigma_\nu = 0 \]
\[ B^\mu_\nu(\lambda_\mathcal{O}) = 0 \]
\[ \dot{B}^\mu_\nu(\lambda_\mathcal{O}) = \delta^\mu_\nu \]

\[ W_{lx}^\mu_\nu = B^\mu_\nu(\lambda_\varepsilon) \]
\[ W_{ul}^\mu_\nu = \dot{B}^\mu_\nu(\lambda_\varepsilon) \]

\[ \delta x^\mu_\varepsilon = W_{xx}^\mu_\nu \delta x^\nu_\mathcal{O} + W_{xl}^\mu_\nu \Delta l^\nu_\mathcal{O} \]
\[ \Delta l^\mu_\varepsilon = W_{lx}^\mu_\nu \delta x^\nu_\mathcal{O} + W_{ll}^\mu_\nu \Delta l^\nu_\mathcal{O} \]

\[ W_{**}: T_\mathcal{O} M \mapsto T_\varepsilon M \]
Light propagation effects

\[ \delta x_\mathcal{O}^\mu \]

\[ \Delta l_\mathcal{O}^\mu = \delta l_\mathcal{O}^\mu + \Gamma^\mu_{\nu\sigma}(\mathcal{O}) l^\nu \delta x_\mathcal{O}^\sigma \]

\[ l + \Delta l_\mathcal{O} \]

\[ \ddot{B}_\mu^\nu - R^\mu_{\alpha\beta\sigma} l^\alpha l^\beta B_\sigma^\nu = 0 \]

\[ B_\mu^\nu(\lambda_\mathcal{O}) = 0 \]

\[ \dot{B}_\mu^\nu(\lambda_\mathcal{O}) = \delta_\mu^\nu \]

\[ W_{lx_\mu^\nu} = B_\mu^\nu(\lambda_\mathcal{E}) \]

\[ W_{ll_\mu^\nu} = \dot{B}_\mu^\nu(\lambda_\mathcal{E}) \]

\[ \delta x_\mathcal{E}^\mu = W_{xx_\mu^\nu} \delta x_\mathcal{O}^\nu + W_{xl_\mu^\nu} \Delta l_\mathcal{O}^\nu \]

\[ \Delta l_\mathcal{E}^\mu = W_{lx_\mu^\nu} \delta x_\mathcal{O}^\nu + W_{ll_\mu^\nu} \Delta l_\mathcal{O}^\nu \]

\[ W^{**} : T_\mathcal{O} M \mapsto T_\mathcal{E} M \]

Bilocal geodesic operators (bitensors)

Light propagation effects

\[ \delta x^\mu_O \]
\[ \Delta l^\mu_O = \delta l^\mu_O + \Gamma^\mu_{\nu\sigma}(O) l^\nu \delta x^\sigma_O \]

\[ l + \Delta l_O \]

\[ \ddot{B}^\mu_\nu - R^\mu_{\alpha\beta\sigma} l^\alpha l^\beta B^\sigma_\nu = 0 \]
\[ B^\mu_\nu(\lambda_O) = 0 \]
\[ \dot{B}^\mu_\nu(\lambda_O) = \delta^\mu_\nu \]
\[ W_{lx}^\mu_\nu = B^\mu_\nu(\lambda_\varepsilon) \]
\[ W_{ll}^\mu_\nu = \dot{B}^\mu_\nu(\lambda_\varepsilon) \]

\[ \delta x^\mu_\varepsilon = W_{xx}^\mu_\nu \delta x^\nu_O + W_{xl}^\mu_\nu \Delta l^\nu_O \]
\[ \Delta l^\mu_\varepsilon = W_{lx}^\mu_\nu \delta x^\nu_O + W_{ll}^\mu_\nu \Delta l^\nu_O \]

\[ W_{**} : T_O M \mapsto T_\varepsilon M \]

Bilocal geodesic operators (bitensors)


Nonlinear functionals of the curvature tensor
Apparent position on the observer’s sky
\( \gamma_0 \) as the reference null geodesic
Observables

Apparent position on the observer’s sky
$\gamma_0$ as the reference null geodesic

- position the sky

$$r^\mu = \frac{l_\phi^\mu + \Delta l_\phi^\mu}{(l_\phi^\sigma + \Delta l_\phi^\sigma) u_\phi^\sigma} + u_\phi^\mu$$

$$\delta r^A = \frac{\Delta l_\phi^A}{(l_\phi^\sigma + \Delta l_\phi^\sigma) u_\phi^\sigma} = \frac{\Delta l_\phi^A}{l_\phi^\sigma u_\phi^\sigma} + O(\Delta l_\phi^2)$$
Observables

Apparent position on the observer’s sky
$\gamma_0$ as the reference null geodesic

- position the sky

$$ r^\mu = \frac{l_\Theta^\mu + \Delta l_\Theta^\mu}{(l_\Theta^\sigma + \Delta l_\Theta^\sigma) u_\Theta^\sigma} + u_\Theta^\mu $$

$$ \delta r^A = \frac{\Delta l_\Theta^A}{(l_\Theta^\sigma + \Delta l_\Theta^\sigma) u_\Theta^\sigma} = \frac{\Delta l_\Theta^A}{l_\Theta^\sigma u_\Theta^\sigma} + O(\Delta l_\Theta^2) $$

Parallel rays approximation (PRA)
Apparent position on the observer’s sky

\( \gamma_0 \) as the reference null geodesic

- **position the sky**

\[
\rho^\mu = \frac{l^\mu_\Omega + \Delta l^\mu_\Omega}{(l^\sigma_0 + \Delta l^\sigma_0) u^\sigma_\sigma} + u^\mu_\Omega
\]

\[
\delta r^A = \frac{\Delta l^A_\Omega}{(l^\sigma_\Omega + \Delta l^\sigma_\Omega) u^\sigma_\sigma} = \frac{\Delta l^A_\Omega}{l^\sigma_\Omega u^\sigma_\sigma} + O(\Delta l^2_\Omega)
\]

**Parallel rays approximation (PRA)**

- **time of arrival**

\[
g_{\mu\nu} (l^\mu_\Omega + \Delta l^\mu_\Omega) (l^\nu_\Omega + \Delta l^\nu_\Omega) = 0
\]

\[
g_{\mu\nu} l^\mu_\Omega \Delta l^\nu_\Omega + O(\Delta l^2_\Omega) = 0
\]
Apparent position on the observer’s sky
\( \gamma_0 \) as the reference null geodesic

- **position the sky**
  \[
  r^\mu = \frac{l_\sigma^\mu + \Delta l_\sigma^\mu}{(l_\sigma^\mu + \Delta l_\sigma^\mu) u_\sigma} + u_\sigma^\mu
  \]

- **time of arrival**
  \[
  g_{\mu\nu} (l_\sigma^\mu + \Delta l_\sigma^\mu) (l_\sigma^\nu + \Delta l_\sigma^\nu) = 0
  \]

**Parallel rays approximation (PRA)**

**Flat light cones approximation (FLA)**
Apparent position on the observer’s sky
\( \gamma_0 \) as the reference null geodesic

### position the sky
\[
{r^\mu} = \frac{l_\varnothing^\mu + \Delta l_\varnothing^\mu}{(l_\varnothing^s + \Delta l_\varnothing^s) u_{\varnothing^s}} + u_\varnothing^\mu
\]
\[
\delta r^A = \frac{\Delta l^A_\varnothing}{(l^s_\varnothing + \Delta l^s_\varnothing) u_{\varnothing^s}} = \frac{\Delta l^A_\varnothing}{l^s_\varnothing u_{\varnothing^s}} + O(\Delta l^2_\varnothing)
\]

**Parallel rays approximation (PRA)**

### time of arrival
\[
g_{\mu\nu} (l_\varnothing^\mu + \Delta l_\varnothing^\mu) (l_\varnothing^\nu + \Delta l_\varnothing^\nu) = 0
\]
\[
g_{\mu\nu} l_\varnothing^\mu \Delta l_\varnothing^\mu + O(\Delta l^2_\varnothing) = 0
\]

**Flat light cones approximation (FLA)**

\[
\delta x_\varnothing^\mu l_{\varnothing^s} = \delta x_\varnothing^\mu l_{\varnothing^s}
\]
Flat light cones approximation

$$\delta x^\mu_{O} l_{O,\mu} = \delta x^\mu_{E} l_{E,\mu}$$

No transverse Rømer delays
Approximations

**Flat light cones approximation**

\[ \delta x^\mu \delta_{\mu} = \delta x^\mu l_{\mu} \]

No transverse Rømer delays

**Parallel rays approximation**

\[ \delta r^A = \frac{\Delta l^A}{l_{\sigma} u_{\sigma}} \]

No perspective distortions

\[ l \] components drop out, only the timelike and transverse remain
Approximations

Flat light cones approximation

$$\delta x^\mu_{\mathcal{O}} l_{\mathcal{O} \mu} = \delta x^\mu_{\mathcal{E}} l_{\mathcal{E} \mu}$$

No transverse Rømer delays

Parallel rays approximation

$$\delta r^A = \frac{\Delta l^A_{\mathcal{O}}}{l^\sigma_{\mathcal{O}} u_{\mathcal{O} \sigma}}$$

No perspective distortions

\(l\) components drop out, only the timelike and transverse remain

Applicability

almost flat: keep \(\frac{L}{R}\), disregard \(\left(\frac{L}{R}\right)^2\)
\[ \delta x^\mu = W_{XX}^\mu_{\nu} \delta x^\nu + W_{XL}^\mu_{\nu} \Delta l^\nu \]

\[ \delta x^\mu l_O^\mu = \delta x^\mu l_O \mu \]
Displacement formulas

\[ \mathcal{D}^A_B \Delta l^B_\Theta = \delta x^A_\Theta - \hat{\delta} x^A_\Theta - m^A_\mu \delta x^\mu_\Theta \]

\[ \delta x^\mu_\Theta l_\Theta \mu = \delta x^\mu l_\Theta \mu \]
Displacement formulas

\[ \mathcal{D}^A_B \Delta l^B = \delta x^A_\Theta - \delta \hat{x}^A_\Theta - m^A_\mu \delta x^\mu_\Theta \]

\[ \delta x^\mu_\Theta l_\Theta^\mu = \delta x^\mu_l l_\Theta^\mu \]

where

\[ ^\wedge = \text{parallel transport} \]
Displacement formulas

\[ \mathcal{D}_B^A \Delta l_\Theta^B = \delta x_\Theta^A - \delta \hat{x}_\Theta^A - m_\mu^A \delta x_\Theta^\mu \]
\[ \delta x_\Theta^\mu l_\Theta^\mu = \delta x_\Theta^\mu l_\Theta^\mu \]

where

\[ ^\wedge = \text{parallel transport} \]

\[ \mathcal{D} : \mathcal{P}_\Theta \rightarrow \mathcal{P}_\mathcal{E} \quad \text{Jacobi operator} \]
\[ \overline{\mathcal{D}}^A_B - R^A_{\mu \nu C} l^\mu l^\nu D^C_B = 0 \]
\[ \mathcal{D}^A_B(\lambda_\Theta) = 0 \]
\[ \overline{\mathcal{D}}^A_B(\lambda_\Theta) = \delta^A_B \]
Displacement formulas

\[ \mathcal{D}^A_B \Delta l^B_\Theta = \delta x^A_\xi - \delta \hat{x}^A_\Theta - m^A_\mu \delta x^\mu_\Theta \]

\[ \delta x^\mu_\Theta l^\mu_\mu = \delta x^\mu_\xi l^\mu_\mu \]

where

^ = parallel transport

\[ \mathcal{D} : \mathcal{P}_\Theta \rightarrow \mathcal{P}_\xi \quad \text{Jacobi operator} \]

\[ \mathcal{D}^A_B - R^A_{\mu \nu C} l^\mu l^\nu D^C_B = 0 \]

\[ \mathcal{D}^A_B(\lambda_\Theta) = 0 \]

\[ \mathcal{D}^A_B(\lambda_\Theta) = \delta^A_B \]

\[ m : \mathcal{Q}_\Theta \rightarrow \mathcal{P}_\xi \quad \text{E/O asymmetry operator} \]

\[ \ddot{m}^A_\sigma - R^A_{\mu \nu C} l^\mu l^\nu m^C_\sigma = R^A_{\mu \nu \sigma} l^\mu l^\nu \]

\[ m^A_\mu(\lambda_\Theta) = 0 \]

\[ \dot{m}^A_\mu(\lambda_\Theta) = 0 \]

vanishes in a flat space!

\[ m|_{\mathcal{P}_\Theta} \equiv m_\perp : \mathcal{P}_\Theta \rightarrow \mathcal{P}_\xi \]
Linear image distortion

\[ D^A_B \Delta i^B_C = \delta x^A_A \]
Linear image distortion

\[ D^A_B \Delta i^B_\mathcal{O} = \delta x^A_\mathcal{E} \]

\[ \delta \theta^A \approx \delta r^A = \frac{1}{u_{\mathcal{O}} l_{\mathcal{O} \sigma}} \mathcal{D}^{-1A}_B \delta x^B_\mathcal{E} \]
Linear image distortion

\[ \Delta i_{\mathcal{O}}^B = \delta x_{\mathcal{E}}^A \]

\[ \delta \theta^A \approx \delta r^A = \frac{1}{u_{\mathcal{O}} l_{\mathcal{O}\sigma}} D^{-1A}_{B} \delta x_{\mathcal{E}}^B \]

magnification matrix \( M^A_B \)
Linear image distortion

\[ \mathcal{D}^A_B \Delta i^B_O = \delta x^A_{\xi} \]

\[ \delta \theta^A \approx \delta r^A = \frac{1}{u_0 l_0} \mathcal{D}^{-1}_B A \delta x^B_{\xi} \]

magnification matrix \( M^A_B \)
Linear image distortion

\[ D^A_B \Delta l^B_\Theta = \delta x^A_\Theta \]

\[ \delta \theta^A \approx \delta r^A = \frac{1}{u^\sigma_\Theta l^\sigma_\Theta} D^{-1A}_B \delta x^B_\Theta \]

magnification matrix \( M^A_B \)

angular diameter distance

\[ D_{\text{ang}} = u^\sigma_\Theta l^\sigma_\Theta \left| \det D^A_B \right|^{1/2} \]
Linear image distortion

\[ D^A_B \Delta l^B_\Theta = \delta x^A_\Theta \]

\[ \delta \theta^A \approx \delta r^A = \frac{1}{u^\sigma_\Theta l^\sigma_\sigma} D^{-1B}_A \delta x^B_\Theta \]

magnification matrix \( M^A_B \)

angular diameter distance

\[ D_{ang} = u^\sigma_\Theta l^\sigma_\sigma \left| \det D^A_B \right|^{1/2} \]

\[ M^A_B \equiv M^A_B \left( R^\mu_{\nu\alpha\beta} \left|_{\gamma_0} \right., u^\mu_\Theta \right) \]

\[ D_{ang} \equiv D_{ang} \left( R^\mu_{\nu\alpha\beta} \left|_{\gamma_0} \right., u^\mu_\Theta \right) \]
\[ \mathcal{D}^A_B \Delta l^B = - \delta \hat{x}^A_{\theta} - m_{\perp}^A_B \delta x^B_{\theta} \]
Parallax

\[ \mathcal{D}^A_B \Delta l^B = -\delta \hat{x}^A_{\theta} - m_{\perp}^A_B \delta x^B_{\theta} \]

\[ \delta \theta^A \approx \delta r^A = -\frac{1}{u^A_{\theta} l_{\theta} \sigma} \mathcal{D}^{-1}_C \left( \delta^C B + m_{\perp}^C_B \right) \delta x^B_{\theta} \]
Parallax

\[ \mathcal{D}^A_B \Delta l^B = -\delta \hat{x}^A_\theta - m^A_B \delta x^B_\theta \]

\[ \delta \theta^A \approx \delta r^A = -\frac{1}{u_\theta l_\sigma} \mathcal{D}^{-1}^A_C \left( \delta^C_B + m^C_B \right) \delta x^B_\theta \]

parallax matrix \( \Pi^A_B \)
Parallax

\[ \mathcal{D}_B^A \Delta l^B = - \delta \hat{x}_\theta^A - m^A_B \delta x^B_\theta \]

\[ \delta \theta^A \approx \delta r^A = - \frac{1}{u_\theta l_\theta \sigma} \mathcal{D}^{-1}_{-1} A C \left( \delta^C_B + m^C_B \right) \delta x^B_\theta \]

parallax matrix \[ \Pi^A_B \]
Parallax

\[ \mathcal{D}_B^A \Delta l^B = - \delta \hat{x}_\theta^A - m_{\perp B}^A \delta x^B \]

\[ \delta \theta^A \approx \delta r^A = - \frac{1}{u_\theta^c l_\theta^c} \mathcal{D}^{-1}_{C}^A \left( \delta C^B + m_{\perp C}^B \right) \delta x^B \]

parallax matrix \[ \Pi^A_B \]

\[ \Pi_{AB} = \Pi_{BA} \]
Parallax

\[ \mathcal{D}_B^A \Delta l^B = -\delta \hat{x}_\theta - m_{\perp}^A B \delta x^B \]

\[ \delta \theta^A \approx \delta r^A = -\frac{1}{u^\sigma l_{\sigma} \theta} \mathcal{D}_C^{-1} B^C \left( \delta C_B + m_{\perp} C_B \right) \delta x^B \]

parallax matrix \[ \Pi^A_B \]

\[ \Pi_{AB} = \Pi_{BA} \]

parallax distance

\[ D_{par} = u^\sigma l_{\sigma} \theta \left| \det \mathcal{D}_B^A \right|^{1/2} \left| \det \left( \delta B^A + m_{\perp}^A B \right) \right|^{-1/2} \]
Parallax

\[ D^A_B \Delta l^B = - \delta \hat{x}_\theta^A - m_{\| A}^A \delta x^B \]

\[ \delta \theta^A \approx \delta r^A = - \frac{1}{u_{\theta}^{\sigma} l_{\sigma}} D^{-1}_C^A \left( \delta^C_B + m_{\perp C}^C \right) \delta x^B \]

Parallax matrix

\[ \Pi^A_B = \Pi_{AB} = \Pi_{BA} \]

Parallax distance

\[ D_{par} = u_{\theta}^{\sigma} l_{\sigma} \left| \det D^A_B \right|^{1/2} \left| \det \left( \delta^A_B + m_{\perp A}^A \right) \right|^{-1/2} \]

\[ D_{par} \equiv D_{par} \left( R^\mu_{\nu \alpha \beta} \bigg\vert_{\gamma_0}, u^\mu_\theta \right) \]
Parallax in a general situation
Parallax in a general situation

Both observer and emitter in bound systems
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects

\[ \delta x_\theta^\mu = U_\theta^\mu t_\theta + \sigma^\mu(t_\theta) \]

\[ \sigma^\mu l_\theta^{-\mu} = 0 \]
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects

\[
\delta x_\Theta^\mu = U_\Theta^\mu t_\Theta + \sigma^\mu(t_\Theta)
\]

\[
\sigma^\mu l_\Theta^\mu = 0
\]

\[
\delta x_\Xi^\mu = U_\Xi^\mu t_\Xi + \rho^\mu(t_\Xi)
\]

\[
\rho^\mu l_\Xi^\mu = 0
\]
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects

\[
\delta x^\mu_\varnothing = U^\mu_\varnothing t_\varnothing + \sigma^\mu(t_\varnothing) \quad \sigma^\mu l_{\varnothing \mu} = 0
\]

\[
\delta x^\mu_\mathcal{E} = U^\mu_\mathcal{E} t_\mathcal{E} + \rho^\mu(t_\mathcal{E}) \quad \rho^\mu l_{\mathcal{E} \mu} = 0
\]

\[
\delta \theta^A = \delta_\varnothing \rho^A t_\varnothing + M^A_B \rho^B \left( (1 + z)^{-1} t_\varnothing \right) - \Pi^A_B \sigma^B(t_\varnothing)
\]
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects

\[
\delta x^\mu_{\theta} = U^\mu_{\theta} t_{\theta} + \sigma^\mu(t_{\theta}) \quad \sigma^\mu l_{\theta \mu} = 0
\]

\[
\delta x^\mu_{\varepsilon} = U^\mu_{\varepsilon} t_{\varepsilon} + \rho^\mu(t_{\varepsilon}) \quad \rho^\mu l_{\varepsilon \mu} = 0
\]

\[
\delta \theta^A = \delta_\theta r^A t_{\theta} + M^A_B \rho^B ((1 + z)^{-1} t_{\theta}) - \Pi^A_B \sigma^B(t_{\theta})
\]

barycenter drift (linear)
Parallax in a general situation

Both observer and emitter in bound systems

Barycenters in free fall

Question: parallax without the aberration effects

\[
\delta x^\mu_\phi = U^\mu_\phi t_\phi + \sigma^\mu(t_\phi)
\]

\[
\delta x^\mu_\xi = U^\mu_\xi t_\xi + \rho^\mu(t_\xi)
\]

\[
\sigma^\mu l^\phi_\mu = 0
\]

\[
\rho^\mu l^\xi_\mu = 0
\]

\[
\delta \theta^A = \delta_\phi^A t_\phi + M^A_B \rho^B \left( (1 + z)^{-1} t_\phi \right) - \Pi^A_B \sigma^B(t_\phi)
\]

barycenter drift (linear)

parallax (periodic)
Motions-independent observables

\[ z \equiv z(u^\mu_\hat{\theta}, u^\mu_\hat{\xi}) \]
\[ r^\mu \equiv r^\mu(u^\mu_\hat{\theta}) \]
\[ M^A_B \equiv M^A_B \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta} \right) \]
\[ \Pi^A_B \equiv \Pi^A_B \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta} \right) \]
\[ D_{\text{ang}} \equiv D_{\text{ang}} \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta} \right) \]
\[ D_{\text{par}} \equiv D_{\text{par}} \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta} \right) \]
\[ \delta_\theta r^A \equiv \delta_\theta r^A \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta}, u^\mu_\hat{\xi}, w^\mu_\hat{\theta} \right) \]
\[ \frac{dz}{d\tau_\theta} \equiv \frac{dz}{d\tau_\theta} \left( R^\mu_{\nu\alpha\beta} \bigg|_{\gamma_0}, u^\mu_\hat{\theta}, u^\mu_\hat{\xi}, w^\mu_\hat{\theta}, w^\mu_\hat{\xi} \right) \]
Motions-independent observables

\[ z \equiv z(u^\mu_\mathcal{O}, u^\mu_\mathcal{E}) \]

\[ r^\mu \equiv r^\mu(u^\mu_\mathcal{O}) \]

\[ M^A_B \equiv M^A_B(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}) \]

\[ \Pi^A_B \equiv \Pi^A_B(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}) \]

\[ D_{\text{ang}} \equiv D_{\text{ang}}(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}) \]

\[ D_{\text{par}} \equiv D_{\text{par}}(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}) \]

\[ \delta_\mathcal{O}r^A \equiv \delta_\mathcal{O}r^A(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}, u^\mu_\mathcal{E}, w^\mu_\mathcal{O}, w^\mu_\mathcal{E}) \]

\[ \frac{dz}{d\tau_\mathcal{O}} \equiv \frac{dz}{d\tau_\mathcal{E}}(R^\mu_{\nu\alpha\beta}, u^\mu_\mathcal{O}, u^\mu_\mathcal{E}, w^\mu_\mathcal{O}, w^\mu_\mathcal{E}) \]
Motions-independent observables

\[ z \equiv z(\omega_{\phi}, \omega_{\phi}) \]

\[ r^\mu \equiv r^\mu(\omega_{\phi}) \]

\[ M^A_B \equiv M^A_B \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi} \right) \]

\[ \Pi^A_B \equiv \Pi^A_B \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi} \right) \]

\[ D_{ang} \equiv D_{ang} \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi} \right) \]

\[ D_{par} \equiv D_{par} \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi} \right) \]

\[ \delta_{\phi} r^A \equiv \delta_{\phi} r^A \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi}, u^\mu_{\phi}, w^\mu_{\phi} \right) \]

\[ \frac{dz}{d\tau_{\phi}} \equiv \frac{dz}{d\tau_{\phi}} \left( R^\mu_{\nu\alpha\beta} | \gamma_0 , u^\mu_{\phi}, u^\mu_{\phi}, w^\mu_{\phi}, w^\mu_{\phi} \right) \]

\[ w^A_B \equiv M^{-1^A_C} \Pi^C_B \]
Motions-independent observables

\[ z \equiv z(u^\mu_\sigma, u^\mu_\xi) \]
\[ r^\mu \equiv r^\mu(u^\mu_\sigma) \]

\[ M^A_B \equiv M^A_B \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma \right) \]
\[ \Pi^A_B \equiv \Pi^A_B \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma \right) \]
\[ D_{ang} \equiv D_{ang} \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma \right) \]
\[ D_{par} \equiv D_{par} \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma \right) \]

\[ \delta_\sigma r^A \equiv \delta_\sigma r^A \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma, u^\mu_\xi, w^\mu_\sigma, w^\mu_\xi \right) \]
\[ \frac{dz}{d\tau_\sigma} \equiv \frac{dz}{d\tau_\sigma} \left( R^\mu_{\nu\alpha\beta} \big|_{\gamma_0}, u^\mu_\sigma, u^\mu_\xi, w^\mu_\sigma, w^\mu_\xi \right) \]

\[ M^A_B = \frac{1}{u^\sigma_\xi l_{\sigma \xi}} D^{-1A}_B \]
\[ \Pi^A_B = \frac{1}{u^\sigma_\xi l_{\sigma \xi}} D^{-1A}_C \left( \delta^C_B + m^C_B \right) \]

\[ w^A_B = M^{-1A}_C \Pi^C_B \]
Motions-independent observables

\[ z \equiv z(u_\phi^\mu, u_\phi^\nu) \]
\[ r^\mu \equiv r^\mu(u_\phi^\mu) \]

\[ M_B^A \equiv M_B^A \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu \right) \]
\[ \Pi_B^A \equiv \Pi_B^A \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu \right) \]
\[ D_{ang} \equiv D_{ang} \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu \right) \]
\[ D_{par} \equiv D_{par} \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu \right) \]
\[ \delta_{\phi} r^A \equiv \delta_{\phi} r^A \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu, u_\phi^\nu, w_\phi^\mu \right) \]
\[ \frac{dz}{d\tau_{\phi}} \equiv \frac{dz}{d\tau_{\phi}} \left( R_\mu^{\nu \alpha \beta}_{\gamma_0}, u_\phi^\mu, u_\phi^\nu, w_\phi^\mu, w_\phi^\nu \right) \]

\[ w_{\perp}^A_B = M^{-1^A}_C \Pi^C_B \]
Motions-independent observables

\[ z \equiv z(u_\mu^\nu \omega, u_\mu^\nu \epsilon) \]

\[ r_\mu \equiv r_\mu(u_\mu^\nu \omega) \]

\[ M^A_B \equiv M^A_B(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega) \]

\[ \Pi^A_B \equiv \Pi^A_B(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega) \]

\[ D_{ang} \equiv D_{ang}(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega) \]

\[ D_{par} \equiv D_{par}(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega) \]

\[ \delta_\theta r^A \equiv \delta_\theta r^A(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega, w_\mu^\nu) \]

\[ \frac{dz}{d\tau_\theta} \equiv \frac{dz}{d\tau_\theta}(R^\mu_{\nu\alpha\beta} \gamma_0, u_\mu^\nu \omega, u_\mu^\nu \epsilon, w_\mu^\nu, w_\mu^\nu) \]

\[ w_{\perp}^A_B = M^{-1}C \pi_{B}^C \]

\[ w_{\perp}^A_B \equiv w_{\perp}^A_B(R^\mu_{\nu\alpha\beta} \gamma_0) = \delta^A_B + m_{\perp}^A_B \]
Parameter $\mu$

$$\mu = 1 - \det w^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B}$$
Parameter $\mu$

$$\mu = 1 - \det w_B^A = 1 - \frac{\det \Pi^A_B}{\det M^A_B}$$

- scalar, dimensionless
Parameter $\mu$

$$\mu = 1 - \det w^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B}$$

- scalar, dimensionless
- momentary motions-independent
Parameter $\mu$

$$\mu = 1 - \det w^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B}$$

- scalar, dimensionless
- momentary motions-independent
- (theoretically) simple to measure

$$\mu = 1 \mp \frac{D_{\text{ang}}^2}{D_{\text{par}}^2}$$
Parameter $\mu$

\[ \mu = 1 - \det w_{\perp A}^B = 1 - \frac{\det \Pi_A^B}{\det M_A^B} \]

- scalar, dimensionless
- momentary motions-independent
- (theoretically) simple to measure

\[ \mu = 1 \mp \frac{D_{\text{ang}}^2}{D_{\text{par}}^2} \]
Parameter $\mu$

\[ \mu = 1 - \det w_A^B = 1 - \frac{\det \Pi^A_B}{\det M^A_B} \]

- scalar, dimensionless
- momentary motions-independent
- (theoretically) simple to measure

\[ \mu = 1 \mp \frac{D_{ang}^2}{D_{par}^2} \]
Parameter $\mu$

$\mu = 1 - \det w_A^B = 1 - \frac{\det \Pi^A_B}{\det M^A_B}$

- scalar, dimensionless
- momentary motions-independent
- (theoretically) simple to measure

$\mu = 1 \mp \frac{D^2_{\text{ang}}}{D^2_{\text{par}}}$

- no need to measure the parallel transport independently

$m_A^B = w_A^B - \delta_A^B$ parallel transport of perpendicular vectors
Parameter $\mu$

\[
\mu = 1 - \det w_{\perp}^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B} = 1 \pm \frac{D^2_{\text{ang}}}{D^2_{\text{par}}}
\]
Parameter \( \mu \)

\[
\mu = 1 - \det w^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B} = 1 \pm \frac{D_{\text{ang}}^2}{D_{\text{par}}^2}
\]

- curvature detector

\[
\mu = 1 - \det \left( \delta^A_B + m^A_B \right)
\]
Parameter $\mu$

\[
\mu = 1 - \det w_{\perp A}^B = 1 - \frac{\det \Pi^A_B}{\det M^A_B} = 1 \mp \frac{D^2_{\text{ang}}}{D^2_{\text{par}}}
\]

- curvature detector

\[
\mu = 1 - \det \left( \delta^A_B + m^A_B \right)
\]

flat space:

\[
D_{\text{ang}} = D_{\text{par}} = D_\theta
\]

\[
\mu = 0
\]
Parameter \( \mu \)

\[
\mu = 1 - \det w^A_B = 1 - \frac{\det \Pi^A_B}{\det M^A_B} = 1 \mp \frac{D^2_{ang}}{D^2_{par}}
\]

- curvature detector

\[
\mu = 1 - \det \left( \delta^A_B + m^A_B \right)
\]

flat space:

\[
D_{ang} = D_{par} = D_{\theta}
\]

\[
\mu = 0
\]

compare the angular deficit formula

\[
\theta_1 + \theta_2 + \theta_3 - \pi = \int K \, d^2A
\]
Parameter $\mu$

- Spacetime geometry $g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$
- Observation and emission points along a null geodesic $\gamma_0(\lambda), \mathcal{O}, \mathcal{E}$
- Curvature along the line of sight $R^\mu_{\nu\alpha\beta}\big|_{\gamma_0}$
- (Null) geodesic deviation equation
- Bilocal geodesic operators $W_{XX}, W_{XL}, W_{LL}, W_{LX}$
- Covariant expressions for observables
- Observables $r^\mu, z, \delta_{\mathcal{O}} r^A, \delta_{\mathcal{E}} z, \ldots$

$g_{\mu\nu} = \text{[flat]} + \text{[curvature corrections]} \left( R^\mu_{\nu\alpha\beta}\big|_{\gamma_0} \right) + h.o.t.$
Parameter $\mu$

- Spacetime geometry: $g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$
- Observation and emission points along a null geodesic: $\gamma^\mu_0(\lambda), \mathcal{O}, \mathcal{E}$
- Curvature along the line of sight: $R^\mu_{\nu\alpha\beta}(x)$
- (null) Geodesic deviation equation:
  \[ g_{\mu\nu} = \text{[flat]} + \text{[curvature corrections]} \left( R^\mu_{\nu\alpha\beta}(x) \right) + h.o.t. \]
- Bilocal geodesic operators: $W_{XX}, W_{XL}, W_{LL}, W_{LX}$
- Covariant expressions for observables
- Observables: $r^\mu, z, \delta_\mathcal{O} r^A, \delta_\mathcal{E} z, \ldots$
Parameter \( \mu \)

Approximate formula for short distances/small curvature

Linearisation in the curvature of all expressions…
Approximate formula for short distances/small curvature

Linearisation in the curvature of all expressions...

\[ \mu \approx - m_{A} \approx - \int_{\lambda_{0}}^{\lambda} R_{\lambda A}^{A} (\lambda) (\lambda_{e} - \lambda) \, d\lambda \]
Parameter $\mu$

Approximate formula for short distances/small curvature

Linearisation in the curvature of all expressions...

$$\mu \approx - m_{\perp}^{A} \approx - \int_{\lambda_0}^{\lambda_G} R_{\perp A}^{A} (\lambda) (\lambda_G - \lambda) \, d\lambda$$

a bit of tensor manipulation…
Approximate formula for short distances/small curvature

Linearisation in the curvature of all expressions…

\[ \mu \approx - m_A^A \approx - \int_{\lambda_0}^{\lambda_g} R^A_{llA}(\lambda) (\lambda_g - \lambda) d\lambda \]

a bit of tensor manipulation…

\[ \mu \approx \frac{8\pi G}{c^4} \int_{\lambda_0}^{\lambda_g} T_{ll}(\lambda) (\lambda_g - \lambda) d\lambda \]
Weighing spacetime along the line of sight

- Tidal forces, gravitational waves (non-local)
- Weyl tensor $C_{\nu\alpha\beta\gamma}^\mu|_{\gamma_0}$
- Ricci tensor $R_{\mu\nu}|_{\gamma_0}$
- Cosmological constant $\Lambda$
- Matter content along the line of sight (local) $T_{\mu\nu}|_{\gamma_0}$
- Einstein equations
- Ricci decomposition
- Curvature along the line of sight $R_{\nu\alpha\beta\gamma}^\mu|_{\gamma_0}$
- (Null) Geodesic deviation equation
- Bilocal geodesic operators $W_{XX}, W_{XL}, W_{LX}$
- Covariant expressions for observables
- Observables $r^\mu, z, \delta_0 r^A, \delta_0 z, \ldots$
- Momentary positions and motions of the observer and emitter $\delta x^\mu_0, \delta x^\mu_\xi, u^\mu_0, u^\mu_\xi, w^\mu_0, w^\mu_\xi$
- Tidal forces, gravitational waves (non-local)
- Weyl tensor $C_{\nu\alpha\beta\gamma}^\mu|_{\gamma_0}$
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- Observables $r^\mu, z, \delta_0 r^A, \delta_0 z, \ldots$
Weighing spacetime along the line of sight

- tidal forces, gravitational waves (non-local)
- Weyl tensor $C_{\nu\lambda\beta}^{\mu}$
- Ricci tensor $R_{\mu\nu}$
- curvature along the line of sight $R_{\nu\lambda\beta}^{\mu}$
- (null) geodesic deviation equation
- bilocal geodesic operators $W_{XX}, W_{XL}, W_{LL}, W_{LX}$
- covariant expressions for observables
- observables $r^{\mu}, z, \delta_{\Omega}r^{A}, \delta_{\Omega}z, \ldots$
- matter content along the line of sight (local) $T_{\mu\nu}\mid_{\gamma_{0}}$
- Einstein equations
- tidal forces, gravitational waves (non-local)
- Weyl tensor $C_{\nu\lambda\beta}^{\mu}$
- Ricci tensor $R_{\mu\nu}$
- curvature along the line of sight $R_{\nu\lambda\beta}^{\mu}$
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- (null) geodesic deviation equation
- momentary positions and motions of the observer and emitter $\delta x^{\mu}_{\Omega}, \delta x^{\mu}_{\xi}, u^{\mu}_{\Omega}, u^{\mu}_{\xi}, w^{\mu}_{\Omega}, w^{\mu}_{\xi}$
- cosmological constant $\Lambda$
- Einstein equations
Weighing spacetime along the line of sight

- tidal forces, gravitational waves (non-local)
- Weyl tensor $C_{\nu\alpha\beta}^{\mu}$
- Ricci tensor $R_{\mu\nu}$
- curvature along the line of sight $R_{\nu\alpha\beta}^{\mu}$
- (null) geodesic deviation equation
- bilocal geodesic operators $W_{XX}, W_{XL}, W_{LL}, W_{LX}$
- covariant expressions for observables
- momentary positions and motions of the observer and emitter $\delta x_{\mu}, \delta x_{\nu}, u_{\mu}, u_{\nu}, w_{\mu}, w_{\nu}$
- observables $r^\mu, z, \delta_r r^A, \delta_r z, \ldots$
- Einstein equations
- cosmological constant $\Lambda$
- matter content along the line of sight (local) $T_{\mu\nu} \big|_{r_0}$
Weighing spacetime along the line of sight

- tidal forces, gravitational waves (non-local)
- Weyl tensor $C^\mu_{\nu\alpha\beta}$
- cosmological constant $\Lambda$
- Ricci tensor $R_{\mu\nu}$
- curvature along the line of sight $R^\mu_{\ 
u\alpha\beta}$
- (null) geodesic deviation equation
- bilocal geodesic operators $W_{XX}, W_{XL}, W_{LL}, W_{LX}$
- covariant expressions for observables
- observables $r^\mu, z, \delta_\Theta r^A, \delta_\Theta z, \ldots$
- matter content along the line of sight (local) $T_{\mu\nu}$
- Einstein equations
- Ricci decomposition
- Einstein equations
- Einstein equations
- Einstein equations
- Einstein equations
- Einstein equations
Weighing spacetime along the line of sight

- tomography-like properties
Weighing spacetime along the line of sight

- tomography-like properties
Weighing spacetime along the line of sight

- tomography-like properties

\[ \mu \approx \frac{8 \pi G}{c^4} \int_{\lambda_0}^{\lambda_\infty} T_{ll}(\lambda) (\lambda_\infty - \lambda) \, d\lambda \]
Parameter $\mu$

Useful?
Useful?

Very small effects (unless for cosmological distances) (E. Villa, MK)
Useful?

Very small effects (unless for cosmological distances) (E. Villa, MK)

On the galactic scale \((D = 30 \text{ kpc}, \rho = 10^{-2} \text{ M}_\odot \text{ pc}^{-3} - 1 \text{ M}_\odot \text{ pc}^{-3})\) \(\mu = 10^{-6} - 10^{-4}\)
Useful?

Very small effects (unless for cosmological distances) (E. Villa, MK)

On the galactic scale \( D = 30 \text{ kpc}, \ \rho = 10^{-2} \text{ M}_\odot \text{ pc}^{-3} - 1 \text{ M}_\odot \text{ pc}^{-3} \) \( \mu = 10^{-6} - 10^{-4} \)

On cosmological scales (MK, E. Villa):

\[
\mu = \frac{3}{2} \Omega_m z^2 + O(z^3)
\]

\( \mu(z = 1) = 0.22 \) \( (\Lambda CDM \ \Omega_{\Lambda_0} = 0.69, \ \Omega_m = 0.31, \ \Omega_k = 0) \)
Parameter $\mu$

Useful?

Very small effects (unless for cosmological distances) (E. Villa, MK)

On the galactic scale ($D = 30 \text{ kpc}, \rho = 10^{-2} \text{ M}_\odot \text{ pc}^{-3} - 1 \text{ M}_\odot \text{ pc}^{-3}$) $\mu = 10^{-6} - 10^{-4}$

On cosmological scales (MK, E. Villa):

$$\mu = \frac{3}{2} \Omega_m z^2 + O(z^3)$$

$$\mu(z = 1) = 0.22 \quad (\LambdaCDM \quad \Omega_{\Lambda_0} = 0.69, \quad \Omega_m = 0.31, \quad \Omega_k = 0)$$

Requires simultaneous measurement of parallax as well as $D_{\text{ang}}$ to an object

Either a standard ruler or a standard candle

$$D_{\text{ang}} = D_{\text{lum}} (1 + z)^{-2}$$
Parameter $\mu$

**Useful?**

Very small effects (unless for cosmological distances) (E. Villa, MK)

On the galactic scale ($D = 30$ kpc, $\rho = 10^{-2} M_\odot$ pc$^{-3}$ - $1 M_\odot$ pc$^{-3}$)  $\mu = 10^{-6} - 10^{-4}$

On cosmological scales (MK, E. Villa):  $\mu = \frac{3}{2} \Omega m_0 z^2 + O(z^3)$

$\mu(z = 1) = 0.22$  $(\Lambda CDM \quad \Omega_{\Lambda_0} = 0.69, \quad \Omega_m = 0.31, \quad \Omega_k = 0)$

Requires simultaneous measurement of parallax as well as $D_{ang}$ to an object

Either a standard ruler or a standard candle

$$D_{ang} = D_{lum} (1 + z)^{-2}$$

Requires many sources to increase the S/N
Parameter $\mu$

Useful?

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Measurements of the annual parallax on cosmological scales impossible today
**Parameter $\mu$**

**Useful?**

Very small effects (unless for cosmological distances) (E. Villa, MK)

On the galactic scale ($D = 30$ kpc, $\rho = 10^{-2}$ $M_\odot$ pc$^{-3}$ - $1$ $M_\odot$ pc$^{-3}$) \( \mu = 10^{-6} - 10^{-4} \)

On cosmological scales (MK, E. Villa): \( \mu = \frac{3}{2} \Omega m_0 z^2 + O(z^3) \)

\( \mu(z = 1) = 0.22 \) \( \) \( (\Lambda CDM \quad \Omega_{\Lambda_0} = 0.69, \quad \Omega_m = 0.31, \quad \Omega_k = 0) \)

Requires simultaneous measurement of parallax as well as $D_{ang}$ to an object

Either a standard ruler or a standard candle

\( D_{ang} = D_{lum} (1 + z)^{-2} \)

Requires *many sources* to increase the S/N

Measurements of the annual parallax on cosmological scales impossible today

…but we may use the motion of the LC wrt CMB frame in the future [Kardashev 1986, Räsänen 2014, Quercellini *et al* 2012, Marcori *et al* 2018], effects borderline visible
Parameter $\mu$
Still:

Conceptually simple

The only observable so insensitive to external gravitational perturbations and peculiar motions (meaning: no systematics due to tidal distortions or peculiar motions!)

Tomography-like measurement
Parameter $\mu$

Still:

Conceptually simple

The only observable so insensitive to external gravitational perturbations and peculiar motions (meaning: no systematics due to tidal distortions or peculiar motions!)

Tomography-like measurement

Similar ideas before:

McCrea 1935 - parallax distance in FLRW metric carries additional information

Weinberg 1970 - parallax distance in FLRW metric determines $k = 0,1,-1$

Kasai 1988, Rosquist 1988 - parallax distance in FLRW (+ perturbations)

Räsänen 2014 - parallax distance vs. luminosity distance as consistency test of FLRW
Summary
New method of curvature determination/weighing matter along the line of sight
New method of curvature determination/weighing matter along the line of sight

Based on gravitational light bending
Summary

New method of curvature determination/weighing matter along the line of sight

Based on gravitational light bending

Uses only astrometry: compare $D_{\text{par}}$ with $D_{\text{ang}}$ (or $D_{\text{lum}}$ and $z$) measured to a single object
New method of curvature determination/weighing matter along the line of sight

Based on gravitational light bending

Uses only astrometry: compare $D_{\text{par}}$ with $D_{\text{ang}}$ (or $D_{\text{lum}}$ and $z$) measured to a single object

Measurement difficult: small effect, parallax determination difficult on long distances
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Measurement difficult: small effect, parallax determination difficult on long distances

Yet: measurement insensitive to the momentary motions or gravitational distortions!
Summary

New method of curvature determination/weighing matter along the line of sight

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Tomography-like, sensitive to dark and baryonic matter along the line of sight
New method of curvature determination/weighing matter along the line of sight

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Tomography-like, sensitive to dark and baryonic matter along the line of sight

Thank you!
Future plans and projects

Cosmological applications of $\mu$ (with E. Villa)

Tests of spacetime isotropy

Local matter density mapping (dark + baryonic)

Determination of cosmological parameters
Future plans and projects

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Redshift-based observables, generalised reciprocity relations

Thank you!
FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A(\Delta l_\theta + C l_\theta) = \delta r^A(\Delta l_\theta)$$

$$(\delta x^\mu_\theta + D l^\mu_\theta) l_\theta \mu = \delta x^\mu_\theta l_\theta \mu$$

Components along $l^\mu = \text{gauge}$
Quotient spaces

FLA & PRA - components proportional to \( l^\mu \) drop out

\[ \delta r^A(\Delta l_\Theta + C l_\Theta) = \delta r^A(\Delta l_\Theta) \]

\[ (\delta x_\Theta^\mu + D l_\Theta^\mu) l_\Theta^\mu = \delta x_\Theta^\mu l_\Theta^\mu \]

Components along \( l^\mu = \) gauge

\[ Q_\Theta = T_\Theta M/l_\Theta \]

\[ [\delta x_\Theta] \in Q_\Theta \]

\[ \dim Q_\Theta = 3 \]
FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A(\Delta l_\theta + C l_\theta) = \delta r^A(\Delta l_\theta)$$

$$(\delta x_\theta^\mu + D l_\theta^\mu) l_\theta\mu = \delta x_\theta^\mu l_\theta\mu$$

Components along $l^\mu = \text{gauge}$

$$Q_\theta = T_\theta M/l_\theta$$

$$[\delta x_\theta] \in Q_\theta$$

$$\dim Q_\theta = 3$$

$$P_\theta = l_\theta^\perp/l_\theta \subset Q_\theta$$

$$[\Delta l_\theta] \in P_\theta$$

$$\dim P_\theta = 2$$
FLA & PRA - components proportional to $l^\mu$ drop out

\[ \delta r^A(\Delta l_\Theta + C l_\Theta) = \delta r^A(\Delta l_\Theta) \]

\[ (\delta x^\mu_\Theta + D l^\mu_\Theta) l_\Theta \mu = \delta x^\mu_\Theta l_\Theta \mu \]

Components along $l^\mu = \text{gauge}$

Forgetting the $l^\mu$ component (gauge)

\[ Q_\Theta = T_\Theta M/l_\Theta \]

\[ [\delta x_\Theta] \in Q_\Theta \]

\[ \dim Q_\Theta = 3 \]

\[ P_\Theta = l_\Theta^\perp/l_\Theta \subset Q_\Theta \]

\[ [\Delta l_\Theta] \in P_\Theta \]

\[ \dim P_\Theta = 2 \]
Quotient spaces

FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A(\Delta l_\Theta + C l_\Theta) = \delta r^A(\Delta l_\Theta)$$

$$(\delta x^\mu_\Theta + D l^\mu_\Theta) l^\mu_\Theta = \delta x^\mu_\Theta l^\mu_\Theta$$

Components along $l^\mu = \text{gauge}$

Forgetting the $l^\mu$ component (gauge)

Helps also with observer invariance of the formalism:

$$Q_\Theta = T_\Theta M/l_\Theta$$

$$[\delta x_\Theta] \in Q_\Theta$$

$$\dim Q_\Theta = 3$$

$$P_\Theta = l^\perp_\Theta/l_\Theta \subset Q_\Theta$$

$$[\Delta l_\Theta] \in P_\Theta$$

$$\dim P_\Theta = 2$$
FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A(\Delta l_\theta + C l_\theta) = \delta r^A(\Delta l_\theta)$$

$$(\delta x^\mu_\theta + D l^\mu_\theta) l^\mu_\theta = \delta x^\mu_\theta l^\mu_\theta$$

Components along $l^\mu = \text{gauge}$

Forgetting the $l^\mu$ component (gauge)

Helps also with observer invariance of the formalism:

$${\mathcal P}_\theta = l_\theta^\perp / l_\theta \subset Q_\theta$$

$$[\Delta l_\theta] \in {\mathcal P}_\theta$$

$$\dim {\mathcal P}_\theta = 2$$

$$Q_\theta = T_\theta M / l_\theta$$

$$[\delta x_\theta] \in Q_\theta$$

$$\dim Q_\theta = 3$$

$${\mathcal P}_\theta$$ inherits a positive-definite metric from $g$
Quotient spaces

FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A(\Delta l_\partial + C l_\partial) = \delta r^A(\Delta l_\partial)$$

$$(\delta x_\partial^\mu + D l_\partial^\mu) l_\partial \mu = \delta x_\partial^\mu l_\partial \mu$$

Components along $l^\mu = \text{gauge}$

Forgetting the $l^\mu$ component (gauge)

Helps also with observer invariance of the formalism:

\[ P_\partial \text{ inherits a positive-definite metric from } g \]

Distances and angles measured by any observer on his or her screen space (Sachs shadow theorem)
Quotient spaces

FLA & PRA - components proportional to $l^\mu$ drop out

$$\delta r^A (\Delta l_\mathcal{O} + C l_\mathcal{O}) = \delta r^A (\Delta l_\mathcal{O})$$

$$(\delta x_\mathcal{O}^\mu + D l_\mathcal{O}^\mu) l_\mathcal{O} \mu = \delta x_\mathcal{O}^\mu l_\mathcal{O} \mu$$

Components along $l^\mu = \text{gauge}$

Forgetting the $l^\mu$ component (gauge)

Helps also with observer invariance of the formalism:

$\mathcal{P}_\mathcal{O}$ inherits a positive-definite metric from $g$

Distances and angles measured by any observer on his or her screen space (Sachs shadow theorem)

Mappings between the quotient spaces have an observer-invariant geometric meaning
Light propagation effects

\[ \delta x^\mu_\mathcal{O} \]

\[ \Delta l^\mu_\mathcal{O} = \delta l^\mu_\mathcal{O} + \Gamma^\mu_{\nu\sigma}(\mathcal{O}) l^\nu \delta x^\sigma_\mathcal{O} \]
\[ \delta x^\mu_O \]
\[ \Delta l^\mu_O = \delta l^\mu_O + \Gamma^\mu_{\nu\sigma}(O) l^\nu \delta x^\sigma_O \]

\[ l + \Delta l_O \]

Can be combined into a single object

\[
\begin{pmatrix}
\delta x^\mu_E \\
\Delta l_{E\nu}
\end{pmatrix}
= \begin{pmatrix}
W_{XX}^\mu\rho & W_{XL}^{\mu\sigma} \\
W_{LX\nu\rho} & W_{LL\nu}^{\sigma}
\end{pmatrix}
\begin{pmatrix}
\delta x^\rho_O \\
\Delta l_{O\sigma}
\end{pmatrix}
\]
Light propagation effects

\[
\delta x_\mathcal{O}^\mu
\]

\[
\Delta l_\mathcal{O}^\mu = \delta l_\mathcal{O}^\mu + \Gamma^{\mu}_{\nu\sigma}(\mathcal{O}) l^\nu \delta x_\mathcal{O}^\sigma
\]

Can be combined into a single object

\[
\begin{pmatrix}
\delta x_\mathcal{E}^\mu \\
\Delta l_\mathcal{E}_\nu
\end{pmatrix}
= \begin{pmatrix}
W_{XX}^{\mu \rho} & W_{XL}^{\mu \sigma} \\
W_{LY}^{\nu \rho} & W_{LL}^{\nu \sigma}
\end{pmatrix}
\begin{pmatrix}
\delta x_\mathcal{O}^\rho \\
\Delta l_\mathcal{O}_{\sigma}
\end{pmatrix}
\]

Symplectic:

\[W^T \Omega W = \Omega\]
Light propagation effects

\[ \delta x^\mu_\mathcal{O} \]
\[ \Delta l^\mu_\mathcal{O} = \delta l^\mu_\mathcal{O} + \Gamma^\mu_{\nu\sigma}(\mathcal{O}) l^{\nu} \delta x^\sigma_\mathcal{O} \]

Can be combined into a single object

\[
\begin{pmatrix}
\delta x^\mu_{\mathcal{E}} \\
\Delta l_{\mathcal{E}_\nu}
\end{pmatrix}
= \begin{pmatrix}
W_{XX}^{\mu}_\rho & W_{X\mathcal{L}L}^{\mu\sigma} \\
W_{LX}^{\rho}_\nu & W_{LL}^{\sigma}_\nu
\end{pmatrix}
\begin{pmatrix}
\delta x^\rho_{\mathcal{O}} \\
\Delta l_{\mathcal{O}\sigma}
\end{pmatrix}
\]

Symplectic: \[ \mathcal{W}^T \Omega \mathcal{W} = \Omega \]

Given by ODE:

\[
\mathcal{W}(\mathcal{O}) = 1 \\
\frac{d}{d\lambda} \mathcal{W} = \begin{pmatrix}
0 & g^{\mu\sigma} \\
R_{\nu\alpha\beta\rho} l^\alpha l^\beta & 0
\end{pmatrix} \mathcal{W}
\]
\[ \mathcal{D}([\Delta l_\Theta]) = [\delta x_\mathcal{E} - \delta \hat{x}_\Theta] - m([\delta x_\Theta]) \]

\[ g([\delta x_\Theta], l_\Theta) = g([\delta x_\mathcal{E}], l_\mathcal{E}) \]
Displacement formulas

\[
\mathcal{D}([\Delta l_\Theta]) = [\delta x_\mathcal{E} - \delta \hat{x}_\Theta] - m([\delta x_\Theta])
\]

\[
g([\delta x_\Theta], l_\Theta) = g([\delta x_\mathcal{E}], l_\mathcal{E})
\]

Two parametrizations of displaced geodesics:

- endpoint positions \([\delta x_\Theta] \quad [\delta x_\mathcal{E}]\)
- initial data \([\delta x_\Theta] \quad [\Delta l_\Theta]\)
Displacement formulas

\[ \mathcal{D}(\Delta l_\Theta) = [\delta x_\mathcal{E} - \delta \hat{x}_\Theta] - m(\delta x_\Theta) \]

\[ g([\delta x_\Theta], l_\Theta) = g([\delta x_\mathcal{E}], l_\mathcal{E}) \]

Two parametrizations of displaced geodesics:

endpoint positions \hspace{1cm} [\delta x_\Theta] \hspace{1cm} [\delta x_\mathcal{E}]

initial data \hspace{1cm} [\delta x_\Theta] \hspace{1cm} [\Delta l_\Theta]

Dimensions:

[\delta x_\Theta] \in \mathcal{Q}_\Theta \Rightarrow 3 \hspace{1cm} [\delta x_\Theta] \in \mathcal{Q}_\Theta \Rightarrow 3

[\delta x_\mathcal{E}] \in \mathcal{Q}_\mathcal{E} \Rightarrow 3 \hspace{1cm} [\Delta l_\Theta] \in \mathcal{P}_\Theta \Rightarrow 2

time lapse condition \hspace{1cm} -1
Displacement formulas

\[ \mathcal{D}([\Delta l_\theta]) = [\delta x_\mathcal{E} - \delta \hat{x}_\theta] - m([\delta x_\theta]) \]

\[ g([\delta x_\theta], l_\theta) = g([\delta x_\mathcal{E}], l_\mathcal{E}) \]

Two parametrizations of displaced geodesics:

- Endpoint positions:
  - Initial data:
    - \([\delta x_\theta] \quad [\delta x_\mathcal{E}]\)
    - \([\delta x_\theta] \quad [\Delta l_\theta] \quad [\delta x_\mathcal{E}] \quad [\Delta l_\mathcal{E}] \quad [\delta x_\mathcal{E}] \quad [\Delta l_\mathcal{E}] \quad [\delta x_\mathcal{E}] \quad [\Delta l_\mathcal{E}] \quad [\delta x_\mathcal{E}] \quad [\Delta l_\mathcal{E}] \]

Dimensions:

- \([\delta x_\theta] \in \Omega_\theta \Rightarrow 3\)
- \([\delta x_\mathcal{E}] \in \Omega_\mathcal{E} \Rightarrow 3\)
- \([\Delta l_\theta] \in \mathcal{P}_\theta \Rightarrow 2\)
- Time lapse condition: \(-1\)

Total dimension = 5
Position drift (proper motion)
Position drift (proper motion)

\[ g([\delta x_\mathcal{O}], l) = g([\delta x_\varepsilon], l) \]

\[ \mathcal{D} ([\Delta l_\mathcal{O}]) = [\delta x_\varepsilon - \delta \dot{x}_\mathcal{O}] - m ([\delta x_\mathcal{O}]) \]
Position drift (proper motion)

\[ g([\delta x_\mathcal{O}], l) = g([\delta x_\mathcal{E}], l) \]

\[ \mathcal{D}([\Delta l_\mathcal{O}]) = [\delta x_\mathcal{E} - \delta \hat{x}_\mathcal{O}] - m([\delta x_\mathcal{O}]) \]

\[
\frac{d\tau_\mathcal{O}}{d\tau_\mathcal{E}} = \frac{l_\mathcal{O} u^\mathcal{O}_\mathcal{E}}{l_\mathcal{O} u^\mathcal{O}_\mathcal{O}} = \frac{1}{1 + z}
\]

\[
\frac{\Delta l^A_\mathcal{O}}{d\tau_\mathcal{O}} = \mathcal{D}^{-1}_B \left( \left( \frac{1}{1 + z} u_\mathcal{E} - \hat{u}_\mathcal{O} \right)^B - m^B_\mu u^\mu_\mathcal{O} \right)
\]
Position drift (proper motion)

\[ g([\delta x_\mathcal{O}], l) = g([\delta x_\mathcal{E}], l) \]

\[ \mathcal{D} ([\Delta l_\mathcal{O}]) = [\delta x_\mathcal{E} - \delta \hat{x}_\mathcal{O}] - m ([\delta x_\mathcal{O}]) \]

\[ \frac{d\tau_\mathcal{O}}{d\tau_\mathcal{E}} = \frac{l_\sigma u_\mathcal{E}^\sigma}{l_\rho u_\mathcal{O}^\rho} = \frac{1}{1 + z} \]

\[ \Delta l_\mathcal{O}^A = \mathcal{D}^{-1}_B A \left( \left( \frac{1}{1 + z} u_\mathcal{E} - \hat{u}_\mathcal{O} \right)^B - m^B_\mu u_\mathcal{O}^\mu \right) \]

\[ \delta_\mathcal{O} r^A = w_\mathcal{O}^A + \frac{1}{l_\sigma u_\mathcal{O}^\sigma} \mathcal{D}^{-1}_B A \left( \left( \frac{1}{1 + z} u_\mathcal{E} - \hat{u}_\mathcal{O} \right)^B - m^B_\mu u_\mathcal{O}^\mu \right) \]
Position drift (proper motion)

\[ g([\delta x_O], l) = g([\delta x_\varepsilon], l) \]

\[ \mathcal{D} ([\Delta l_O]) = [\delta x_\varepsilon - \delta \hat{x}_O] - m ([\delta x_O]) \]

\[
\frac{d\tau_O}{d\tau_\varepsilon} = \frac{l_\sigma u_\sigma^\varepsilon}{l_\rho u_\rho^O} = \frac{1}{1 + z}
\]

\[
\frac{\Delta l_O^A}{d\tau_O} = \mathcal{D}^{-1}_B A \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_O \right)^B - m^B_\mu u_\mu^O \right)
\]

\[
\delta_O r^A = w_O^A + \frac{1}{l_\sigma u_\sigma^O} \mathcal{D}^{-1}_B A \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_O \right)^B - m^B_\mu u_\mu^O \right)
\]

An exact formula for the position drift rate in any curved space
Position drift (proper motion)

\[ g([\delta x_O], l) = g([\delta x_\varepsilon], l) \]

\[ \mathcal{D} ([\Delta l_O]) = [\delta x_\varepsilon - \delta \hat{x}_O] - m ([\delta x_O]) \]

\[ \frac{d\tau_O}{d\tau_\varepsilon} = \frac{l_\sigma u_\varepsilon^\sigma}{l_\rho u_\rho^\rho} = \frac{1}{1 + z} \]

\[ \frac{\Delta l_O^A}{d\tau_O} = \mathcal{D}^{-1}_{AB} \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_O \right)^B - m^B_{\mu} u_\mu^\mu \right) \]

\[ \delta_O r^A = w_O^A + \frac{1}{l_\sigma u_\sigma^\sigma} \mathcal{D}^{-1}_{AB} \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_O \right)^B - m^B_{\mu} u_\mu^\mu \right) \]

An exact formula for the position drift rate in any curved space

Data: curvature along the line of sight + kinematical variables
Position drift (proper motion)

\[ g([\delta x_\mathcal{O}], l) = g([\delta x_\varepsilon], l) \]

\[ \mathcal{D} ([\Delta l_\mathcal{O}]) = [\delta x_\varepsilon - \delta \hat{x}_\mathcal{O}] - m ([\delta x_\mathcal{O}]) \]

\[
\frac{d\tau_\mathcal{O}}{d\tau_\varepsilon} = \frac{l_\sigma u^\sigma_\mathcal{O}}{l_\rho u^\rho_\mathcal{O}} = \frac{1}{1 + z} \]

\[
\frac{\Delta l^A_\mathcal{O}}{d\tau_\mathcal{O}} = \mathcal{D}^{-1}_B \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_\mathcal{O} \right)^B - m^B_{\mu} u^\mu_\mathcal{O} \right) \]

\[
\delta_\mathcal{O} r^A = w^A_\mathcal{O} + \frac{1}{l_\sigma u^\sigma_\mathcal{O}} \mathcal{D}^{-1}_B \left( \left( \frac{1}{1 + z} u_\varepsilon - \hat{u}_\mathcal{O} \right)^B - m^B_{\mu} u^\mu_\mathcal{O} \right) \]

An exact formula for the position drift rate in any curved space

Data: curvature along the line of sight + kinematical variables

Interesting physical consequences…
Position drift (proper motion)

\[ \delta \theta^A r^A = w^A_{\theta} + \frac{1}{l_{\theta \sigma} u^\sigma_{\theta}} \mathcal{D}^{-1} B \left( \frac{1}{1+z} u_\xi - \hat{u}_\theta \right)^B - m^B_{\mu} u^\mu_{\theta} \]
\[ \delta \epsilon r^A = w^A_\epsilon + \frac{1}{l_\theta \sigma u^\sigma_\theta} \mathcal{D}^{-1}_B^A \left( \left( \frac{1}{1 + z} u_\mathcal{E} - \mathbf{\hat{u}}_\theta \right)_B^B - m^B_\mu u^\mu_\theta \right) \]

automatically orthogonal to \( l_\mathcal{E} \)

\[ \in \mathcal{P}_\mathcal{E} \]
Position drift (proper motion)

\[ \delta_{\theta} r^A = w^A_{\theta} + \frac{1}{l_{\theta \sigma} u^\sigma_{\theta}} D^{-1}_B \left( \left( \frac{1}{1 + z} u_{\theta} - \hat{u}_{\theta} \right)^B - m^B_\mu u^\mu_{\theta} \right) \]

\[ \delta_{\theta} r^A = w^A_{\theta} + \frac{1}{D_{\theta}} \left( \frac{1}{1 + z} u_{\theta} - u_{\theta} \right)^A \]

Compare with the flat case

automatically orthogonal to \( l_\mathcal{E} \)

\[ \in \mathcal{P}_\mathcal{E} \]
Position drift (proper motion)

\[ \delta_{\theta}r^A = \psi_{\theta}^A + \frac{1}{l_{\theta} u_{\theta}} D^{-1}_{B} B \left( \left( \frac{1}{1 + z} u_{\phi} - \hat{u}_{\theta} \right) \right)^B - m^B_{\mu} u^{\mu}_{\theta} \]

automatically orthogonal to \( l_{\phi} \)

\[ \in \mathcal{P}_{\phi} \]

Compare with the flat case

\[ \delta_{\theta}r^A = \psi_{\theta}^A + \frac{1}{D_{\theta}} \left( \frac{1}{1 + z} u_{\phi} - u_{\theta} \right)^A \]

- Apparent superluminal motions
Position drift (proper motion)

\[ \delta_{\theta}r^{A} = w^{A}_{\theta} + \frac{1}{l_{\theta}^{\sigma}u^{\sigma}_{\theta}} D^{-1}_{B} \left( \left( \frac{1}{1 + z} u^{B}_{\theta} - \hat{u}^{B}_{\theta} \right)^{B} - m^{B}_{\mu} u^{\mu}_{\theta} \right) \]

automatically orthogonal to \( l_{\theta} \) ∈ \( \mathcal{P}_{\theta} \)

Compare with the flat case

\[ \delta_{\theta}r^{A} = w^{A}_{\theta} + \frac{1}{D_{\theta}} \left( \frac{1}{1 + z} u^{A}_{\theta} - u^{A}_{\theta} \right) \]

- Apparent superluminal motions

\[ \frac{1}{1 + z} \] large (strong blueshift)
Position drift (proper motion)

\[ \delta_\theta r^A = w^{A}_\theta + \frac{1}{l^{\sigma}_{\theta} u^{\sigma}_{\theta}} D^{-1}_{B} \left( \left( \frac{1}{1 + z} u_{\mathcal{E}} - \hat{u}_{\theta} \right)^{B} - m_{B \mu} u^{\mu}_{\theta} \right) \]

automatically orthogonal to \( l_{\mathcal{E}} \) in \( \mathcal{P}_{\mathcal{E}} \)

Compare with the flat case

\[ \delta_\theta r^A = w^{A}_\theta + \frac{1}{D_{\theta}} \left( \frac{1}{1 + z} u_{\mathcal{E}} - u_{\theta} \right)^A \]

- Apparent superluminal motions

\[ \frac{1}{1 + z} \]

large (strong blueshift)

transverse velocity difference
Position drift (proper motion)

**aberration drift**  
**lensing**  
**transverse 4-velocity difference**  
**E-O asymmetry operator**

\[
\delta r^A = w^A + \frac{1}{l_\sigma u^\sigma} \mathcal{D}^{-1} B \left( \left( \frac{1}{1 + z} u^\nu - \hat{u}^\nu \right)^B - m^B_{\mu} u^\mu \right)
\]

automatically orthogonal to \( l^\nu \in \mathcal{P}^\nu \)

Compare with the flat case

\[
\delta r^A = w^A + \frac{1}{D_\sigma} \left( \frac{1}{1 + z} u^\nu - u^\nu \right)^A
\]

- Apparent superluminal motions

\[
\frac{1}{1 + z} \text{ large (strong blueshift)} \Rightarrow \delta r^A \text{ large}
\]

transverse velocity difference

28
\[ \delta_\theta r^A = w_\theta^A + \frac{1}{l_\theta \sigma u_\theta^\sigma} D^{-1A}_B \left( \left( \frac{1}{1 + z} u_\xi - \hat{u}_\theta \right)^B - m^B_\mu u_\theta^\mu \right) \]
\[
\delta_\theta r^A = w^A_\theta + \frac{1}{l_\theta \sigma u^\sigma_\theta} D^{-1}_B \left( \left( \frac{1}{1 + z} u^\sigma - \hat{u}^\sigma_\theta \right)^B - m^B_\mu u^\mu_\theta \right)
\]

- Relation between lensing and position drift
Position drift (proper motion)

\[
\delta_{\phi}r^A = w^A_{\phi} + \frac{1}{l_{\phi\sigma}u_{\phi}^\sigma} \mathcal{D}^{1A}_{B} \left( \left( \frac{1}{1 + z} u_{\epsilon} - \hat{u}_{\phi} \right)^B - m^B_{\mu} u_{\phi}^\mu \right)
\]

- Relation between lensing and position drift
  
  strong magnification = faster position drift rates
Position drift (proper motion)

\[ \delta_{\theta} r^A = w^A_{\theta} + \frac{1}{l_{\theta \sigma} u^\sigma_{\theta}} \mathcal{D}^{-1}_B \left( \left( \frac{1}{1 + z} u_\xi - \hat{u}_\theta \right)^B - m^B_{\mu} u^\mu_{\theta} \right) \]

- Relation between lensing and position drift
  
  strong magnification = faster position drift rates

- Close to a caustic
Position drift (proper motion)

\[ \delta_{\phi} r^A = w^A_{\phi} + \frac{1}{l_{\phi \phi} u^\phi_{\phi}} D^{-1A}_B \left( \left( \frac{1}{1 + \frac{u_\phi}{u_\phi}} - \hat{u}_\phi \right)^B - m^{B \mu} u^\mu_{\phi} \right) \]

- Relation between lensing and position drift
  
  strong magnification = faster position drift rates

- Close to a caustic

\[ D^A_B \text{ becomes degenerate} \]
Position drift (proper motion)

\[ \delta_\theta r^A = w^A_\theta + \frac{1}{l_{\theta \sigma} u^\sigma_\theta} \mathcal{D}^{-1}_B A \left( \left( \frac{1}{1 + z} u_\xi - \hat{u}_\theta \right)^B - m^B_\mu u^\mu_\theta \right) \]

- Relation between lensing and position drift
  
  strong magnification = faster position drift rates

- Close to a caustic

\[ \mathcal{D}^A_B \text{ becomes degenerate} \]

\[ M^A_B \text{ diverges (infinite magnification)} \]
Position drift (proper motion)

\[ \delta_\theta r^A = w^A_\theta + \frac{1}{l_\theta u_\theta} \mathcal{D}^{-1}_B \left( \left( \frac{1}{1 + z} u_\xi - \hat{u}_\theta \right)^B - m^B_\mu u^\mu_\theta \right) \]

- Relation between lensing and position drift
  - strong magnification = faster position drift rates
- Close to a caustic
  - \( \mathcal{D}^A_B \) becomes degenerate
  - \( M^A_B \) diverges (infinite magnification)
  - \( II^A_B \) diverges (infinite parallax)
Position drift (proper motion)

\[ \delta_\theta r^A = w^A_\theta + \frac{1}{l^\theta \sigma u^\sigma_\theta} D^{-1}_B \left( \left( \frac{1}{1 + z} u_\xi - \hat{u}_\theta \right)_B - m^B_{\mu} u^\mu_\theta \right) \]

- Relation between lensing and position drift
  strong magnification = faster position drift rates

- Close to a caustic

  \( D^A_B \) becomes degenerate

  \( M^A_B \) diverges (infinite magnification)

  \( II^A_B \) diverges (infinite parallax)

  \( \delta_\theta r^A \) diverges (infinite drift rate)
Position drift (proper motion)

$$\delta_\Theta r^A = w^A_\Theta + \frac{1}{l^\Theta_\sigma w^\sigma_\Theta} D^{-1A}_B \left( \left( \frac{1}{1+z} u^\xi - \hat{u}_\Theta \right)^B - m^B_\mu u^\mu_\Theta \right)$$

- Relation between lensing and position drift
  
  strong magnification = faster position drift rates

- Close to a caustic

  $D^A_B$ becomes degenerate

  $M^A_B$ diverges (infinite magnification)

  $\Pi^A_B$ diverges (infinite parallax)

  $\delta_\Theta r^A$ diverges (infinite drift rate)

  caustics = position drift and parallax magnifiers
$g_{\mu \nu} = \text{[flat]} + \text{[curvature corrections]} \left( R^\mu_{\nu \alpha \beta} \right)_{f_0} + h.o.t.$
Observables

- Spacetime geometry: $g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$
- Observation and emission points along a null geodesic: $\gamma^\mu_0(\lambda), \mathcal{O}, \mathcal{E}$
- Curvature along the line of sight: $R^\mu_{\nu\alpha\beta}_{\gamma_0}$
- (Null) geodesic deviation equation
- Bilocal geodesic operators: $W_{XX}, W_{XL}, W_{LL}, W_{LX}, \mathcal{D}, m$
- Covariant expressions for observables
- Observables: $r^\mu, z, \delta^A, \delta z, \ldots$

$$g_{\mu\nu} = \text{[flat]} + \text{[curvature corrections]} \left( R^\mu_{\nu\alpha\beta}_{\gamma_0} \right) + h.o.t.$$
Relation to the Sachs formalism
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Sachs 1961, Ehlers, Jordan, Sachs 1961 - formalism based on congruences of null geodesics
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congruence = 1 geodesic through each point = 3-parameter family
Relation to the Sachs formalism

Sachs 1961, Ehlers, Jordan, Sachs 1961 - formalism based on congruences of null geodesics

congruence = 1 geodesic through each point = 3-parameter family

local tensorial quantities $\theta, \sigma_{AB}, \omega_{AB}$
Relation to the Sachs formalism

Sachs 1961, Ehlers, Jordan, Sachs 1961 - formalism based on congruences of null geodesics

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nonlinear, 1st order ODE’s for them
Relation to the Sachs formalism

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nonlinear, 1st order ODE’s for them

equivalent to the BGO formalism
Relation to the Sachs formalism


Congruence = 1 geodesic through each point = 3-parameter family.

Local tensorial quantities $\theta, \sigma_{AB}, \omega_{AB}$.

Nonlinear, 1st order ODE's for them.

Equivalent to the BGO formalism.

Avoids bitensors, BUT...
Sachs 1961, Ehlers, Jordan, Sachs 1961 - formalism based on congruences of null geodesics

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less flexible (need to fix the initial condition for the congruence, cannot change later)
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Relation to the Sachs formalism

Sachs 1961, Ehlers, Jordan, Sachs 1961 - formalism based on congruences of null geodesics

congruence = 1 geodesic through each point = 3-parameter family

local tensorial quantities $\theta, \sigma_{AB}, \omega_{AB}$

nonlinear, 1st order ODE's for them

equivalent to the BGO formalism

avoids bitensors, BUT...

less flexible (need to fix the initial condition for the congruence, cannot change later)

cannot simultaneously displace both endpoints of geodesics

less useful to study drifts and parallax (although Räsänen 2014, Rosquist 1988...
Problem

Light propagation in the geometric optics approximation
Problem

Light propagation in the geometric optics approximation

Observables:
Problem

Light propagation in the geometric optics approximation

Observables:

• time of arrival $\tau_\mathcal{O}(\tau_\mathcal{E})$
Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_\Theta(\tau_\Phi) \)
- **position the sky**
  \[
  r^\mu = \frac{l_\Theta^\mu l_\Theta}{l_\Theta u_{\Theta \sigma}} + u_\Theta^\mu
  \]
Problem

Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_\mathcal{O}(\tau_\mathcal{E}) \)
- **position the sky**
  \[ r^\mu = \frac{l_\mathcal{O}^\mu}{l_\mathcal{O}^\sigma u_\mathcal{O}_\sigma} + u_\mathcal{O}^\mu \]
- **image distortion**
  \[ \delta \theta^A \approx \delta r^A \]
Problem

Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_S(\tau_S) \)
- **position the sky** \( r^\mu = \frac{l^\mu_S}{l^\sigma_S u^\sigma_S} + u^\mu_S \)
- **image distortion** \( \delta \theta^A \approx \delta r^A \)
- **parallax** \( \delta \theta^A \approx \delta r^A \)
Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_\odot(\tau_\odot) \)
- **position the sky** \( r^\mu = \frac{l^\mu_\odot}{l^\sigma_\odot u^\sigma_\odot} + u^\mu_\odot \)
- **image distortion** \( \delta \theta^A \approx \delta r^A \)
- **parallax** \( \delta \theta^A \approx \delta r^A \)
- **position drift** \( \nabla_{u^\sigma} r^\mu \)
   or \( \delta_\odot r^A = \delta_{FW} r^A \)
Problem

Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_\odot(\tau_\odot) \)
- **position the sky** \( r^\mu = \frac{l^\mu_\odot}{l_\odot u_\odot} + u^\mu_\odot \)
- **image distortion** \( \delta\theta^A \approx \delta r^A \)
- **parallax** \( \delta\theta^A \approx \delta r^A \)
- **position drift** \( \nabla_{u_\odot} r^\mu \)
  or \( \delta_\odot r^A = \delta_{FW} r^A \)

Fermi-Walker derivative
\[
\delta_{FW} X^\mu = \nabla_{u_\odot} X^\mu + \left( - w_\odot \nu u_\odot^\mu + u_\odot \nu w_\odot^\mu \right) X^\nu
\]
Problem

Light propagation in the geometric optics approximation

Observables:

- **time of arrival** \( \tau_\Theta(\tau_\xi) \)
- **position the sky** \( r^\mu = \frac{l^\mu_\Theta}{l^\sigma_\Theta u^{\sigma}_\Theta} + u^\mu_\Theta \)
- **image distortion** \( \delta \theta^A \approx \delta r^A \)
- **parallax** \( \delta \theta^A \approx \delta r^A \)
- **position drift** \( \nabla_{u_\Theta} r^\mu \)
  or \( \delta_\Theta r^A = \delta_{FW} r^A \)
- **redshift** \( z = \frac{E_\xi}{E_\Theta} - 1 \)

\[ \delta_{FW} X^\mu = \nabla_{u_\Theta} X^\mu + \left( -w^{\mu}_\Theta u_\Theta + u_\Theta u^{\nu}_\Theta w^\nu_\Theta \right) X^\nu \]
Problem

Light propagation in the geometric optics approximation

Observables:
- **time of arrival** \( \tau_\varnothing(\tau_\varnothing) \)
- **position the sky** \( r^\mu = \frac{l^\mu_\varnothing}{l_\varnothing u^\varnothing_\sigma} + u^\mu_\varnothing \)
- **image distortion** \( \delta \theta^A \approx \delta r^A \)
- **parallax** \( \delta \theta^A \approx \delta r^A \)
- **position drift** \( \nabla_{u_\varnothing} r^\mu \)
  or \( \delta_{\varnothing} r^A = \delta_{FW} r^A \)
- **redshift** \( z = \frac{E_\varnothing}{E_\varnothing} - 1 \)
- **redshift drift** \( \frac{d}{d\tau_\varnothing} z \)

\[ \delta_{FW} X^\mu = \nabla_{u_\varnothing} X^\mu + \left( -w_{\varnothing\nu} u_\varnothing^\nu + u_{\varnothing\nu} w_\varnothing^\nu \right) X^\nu \]

Fermi-Walker derivative
"Naive" approach

Redshift and position drift
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for $z$
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for $z$
- (Time-dependent) Shapiro and geometric delays
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for \( z \)
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of E and O
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for \( z \)
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of E and O
- (Time-dependent) aberration effects
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for $z$
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of $E$ and $O$
- (Time-dependent) aberration effects
- (Time-dependent) gravitational redshift
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for $z$
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of E and O
- (Time-dependent) aberration effects
- (Time-dependent) gravitational redshift
- (Time-dependent) Doppler effect
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for $z$
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of E and O
- (Time-dependent) aberration effects
- (Time-dependent) gravitational redshift
- (Time-dependent) Doppler effect
- ...
"Naive" approach

Redshift and position drift

- Gravitational interactions between bodies
- (Time-dependent) light ray bending
- (Time-dependent) ISW effect for \( z \)
- (Time-dependent) Shapiro and geometric delays
- (Time-dependent) motions of \( E \) and \( O \)
- (Time-dependent) aberration effects
- (Time-dependent) gravitational redshift
- (Time-dependent) Doppler effect
- ...

Very complicated problem!
"Naive" approach

- Spacetime geometry: $g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$
- Geodesic equation
- Null geodesics: $x^\mu(\lambda)$
- Nearby null geodesics
- O’s and E’s worldlines: $\chi^\mu(\tau_\sigma), \chi^\mu(\tau_\epsilon)$
- Time of observation: $\tau_\sigma$
- Observables: $r^\mu, z, \delta_\sigma r^A, \delta_\sigma z, \ldots$
"Naive" approach

spacetime geometry $g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$

geodesic equation

null geodesics $x^\mu(\lambda)$

O’s and E’s worldlines $x^\mu(\tau_{O}), x^\mu(\tau_{E})$

everything depends on $g$ in a complicated way

observables $r^\mu, z, \delta_{\theta} r^A, \delta_{\theta} z, \ldots$

time of observation $\tau_{O}$

subject to nearby null geodesics

No general formulas or relations
Geometric approach

$g_{\mu\nu}(x), \Gamma^\mu_{\alpha\beta}(x)$
Normal coordinates around a point

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^{\alpha} y^{\beta} + O(y^3) \]
Geometric approach

Normal coordinates around a point

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta + O(y^3)$$
Geometric approach

Normal coordinates around a point

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta + O(y^3) \]

Fermi normal coordinates around a geodesic \((\lambda, y^i)\)

\[ g_{\mu\nu} = \eta_{\mu\nu} + C(\mu, \nu) \cdot R_{\mu\nu i j}(\lambda) y^i y^j + O(y^3) \]
Normal coordinates around a point

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta + O(y^3) \]

Fermi normal coordinates around a geodesic \((\lambda, y^i)\)

\[ g_{\mu\nu} = \eta_{\mu\nu} + C(\mu, \nu) \cdot R_{\mu i j}(\lambda) y^i y^j + O(y^3) \]
Normal coordinates around a point

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Fermi normal coordinates around a geodesic \((\lambda, y^i)\)

\[ g_{\mu\nu} = \eta_{\mu\nu} + C(\mu, \nu) \cdot R_{\mu i j}(\lambda) y^i y^j + O(y^3) \]

Geodesics nearby given by the GDE

\[ \nabla_l \nabla_l \xi^\mu - R^\mu_{\nu \alpha \beta}(\lambda) l^\nu l^\alpha \xi^\beta = 0 \]
Geometric approach

Normal coordinates around a point

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^\alpha y^\beta + O(y^3) \]

Fermi normal coordinates around a geodesic \((\lambda, y^i)\)

\[ g_{\mu\nu} = \eta_{\mu\nu} + C(\mu, \nu) \cdot R_{\mu\nu ij}(\lambda) y^i y^j + O(y^3) \]

Geodesics nearby given by the GDE

\[ \nabla_l \nabla_l \xi^\mu - R^\mu_{\nu\alpha\beta}(\lambda) l^\nu l^\alpha \xi^\beta = 0 \]
Normal coordinates around a point

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} y^{\alpha} y^{\beta} + O(y^3) \]

Fermi normal coordinates around a geodesic \((\lambda, y^i)\)

\[ g_{\mu\nu} = \eta_{\mu\nu} + C(\mu, \nu) \cdot R_{\mu i\nu j}(\lambda) y^i y^j + O(y^3) \]

Geodesics nearby given by the GDE

\[ \nabla_l \nabla_l \xi^\mu - R^\mu_{\nu\alpha\beta}(\lambda) \ n^\nu \ n^\alpha \xi^\beta = 0 \]

Synge 1960, Bażański 1977, Alexandrov&Piragas 1979
Vines 2015, Puetzfeld&Obukhov 2016, Uzun 2018
Geometric approach

Redshift and position drift
Geometric approach

Redshift and position drift
Geometric approach

Redshift and position drift

Impact of the spacetime geometry
only via curvature along the line of sight