Definition of holonomies in the model of Causal Dynamical Triangulations

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Plan of talk

- Holonomies in a Euclidean Riemannian manifold with a fractal geometry.
- Model of Causal Dynamical Triangulations as a statistical model of simplicial manifolds.
- Analog of Wilson loops on a simplicial manifold - search for a new type of geometric observables.
- Large-scale Wilson loops and their invariant distribution.

Based on the work with Jan Ambjørn, Anrzej Görlich, JJ and Renate Loll, Phys.Rev. D92 (2015) 024013, ArXiv:1504.01065 [gr-qc]

Holonomies in gravity

The Levi-Civita connection $\Gamma^{\kappa}_{\mu\nu}(x)$ of a Riemannian manifold M with metric $g_{\mu\nu}(x)$ defines a notion of parallel transport of a vector V^μ along a curve $\gamma^\mu(\lambda)$ parametrized by λ.

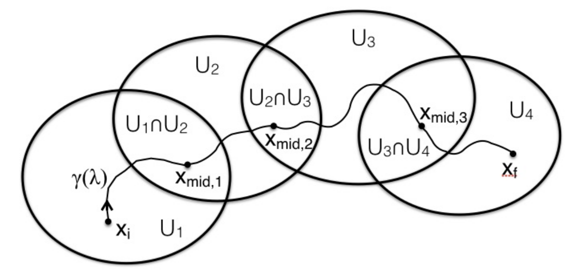
$$V^\mu(x_f) = \left(\mathcal{P}e^{-\int_{\lambda_i}^{\lambda_f}\Gamma_\kappa\dot{\gamma}(\lambda)d\lambda}\right)^{\mu}{}_{\nu} V^\nu(x_i), \quad (\Gamma_\kappa)^\mu{}_\nu = \Gamma^\mu_{\kappa\nu}$$

Under a coordinate transformation $x \to \tilde{x}(x)$ with $M^\mu{}_\nu = \frac{d\tilde{x}^\mu(x)}{dx^\nu}$ and $\dot{\gamma} = \frac{d\gamma}{d\lambda}$, the path-ordered integral transforms non-trivially at the endpoints in accordance with the behavior of vectors

$$\left(\mathcal{P}e^{-\int_{\lambda_i}^{\lambda_f}\Gamma_\kappa\dot{\gamma}(\lambda)d\lambda}\right)^{\mu}{}_{\nu} = M^\mu{}_\alpha(x_f)\left(\mathcal{P}e^{-\int_{\lambda_i}^{\lambda_f}\Gamma_\kappa\dot{\gamma}(\lambda)d\lambda}\right)^{\alpha}{}_{\beta}(M^{-1}(x_i))^\beta{}_\nu$$

Constructing path-ordered product

Let us consider a path $\gamma(\lambda)$ which passes through a sequence of of coordinate neighborhoods U_k with (usually different) coordinates x_k^{μ}. This would be a typical situation on a fractal manifold.



From U_1 to U_2

- Consider a simple example of a path between two open neighborhoods U_1 and U_2 with a non-empty intersection $U_1 \cap U_2$. Let $x_1 \in U_1$, $x_2 \in U_2$ and $x_{mid} \in U_1 \cap U_2$.
- For x_{mid} we have two possible coordinate system $x_{mid,1}$ and $x_{mid,2}$ with $M(x_{mid})^{\mu}{}_{\nu} = \frac{\partial x_2^{\mu}}{\partial x_1^{\nu}}|_{x_{mid}}$.
- Let the corresponding Levi-Civita connections be Γ_1 and Γ_2.
- Connecting the paths in U_1 and U_2 we get

$$\left(\mathcal{P}e^{-\int_{x_{mid}}^{x_2}\Gamma_2}\right)^{\mu}{}_{\nu} M(x_{mid})^{\nu}{}_{\lambda} \left(\mathcal{P}e^{-\int_{x_1}^{x_{mid}}\Gamma_1}\right)^{\lambda}{}_{\kappa}$$

From U_1 to U_n

This expression can be easily generalized to any number of coordinate patches needed to form a holonomy loop from x_1 to x_n

$$\prod_{k=2}^{n}\left(\mathcal{P}e^{-\int_{x_{mid,k-1}}^{x_k}\Gamma_k}\cdot M(x_{mid,k})\right)\cdot\left(\mathcal{P}e^{-\int_{x_1}^{x_{mid,1}}\Gamma_1}\right)$$

where $M(x_{mid,k}) = \frac{\partial x_k(x_{k-1})}{\partial x_{k-1}}|_{x_{mid,k}}$ and "$\cdot$" is a matrix multiplication.

It can be easily shown that this product does not depend on a precise choice of $x_{mid,k} \in U_k \cap U_{k-1}$

Wilson loops

By a Wilson loop one usually means a parallel transport over a closed curve starting and finishing at x_1. It follows that under a change of coordinates $x_1 \to \tilde{x}_1$ such a loop transforms locally

$$\left(\mathcal{P}e^{-\oint \Gamma_1}\right)_{\tilde{x}_1} = M(x_1)\left(\mathcal{P}e^{-\oint \Gamma_1}\right)_{x_1} M^{-1}(x_1), \quad M(x) = \frac{\partial \tilde{x}}{\partial x}$$

It follows that the conjugacy class of a holonomy matrix $\mathcal{P}e^{-\oint \Gamma_1}$ is **coordinate independent.** The only remaining dependence is that on the shape of the loop $\gamma(\lambda)$.

Wilson loops cont.

- Gravitational Wilson loops were little studied in the perturbative quantum gravity.
- The analog of Wilson loops is well known in the Loop Quantum Gravity, where the nonlocal holonomies along spatial curves are a part of the set of fundamental variables. They are promoted to finite operators, assumed to not need any renormalization.
- They are different than the loops in the ordinary gauge theory, where the expectation values of Wilson loops need to be renormalized.
- The discussion below is the attempt to use the analogy with QCD in the model of Causal Dynamical Triangulations, a model in some sense similar to a lattice formulation of QCD.

What is CDT

Attempt to formulate quantum theory of a 3d geometry.

- Spatial geometry is treated as a quantum object.
- Following the ideas by Regge we define a spatial state as a discretization of a 3d geometry with a (closed) topology Σ. It is constructed using regular simplices with a common edge length a_s.
- States evolve in a (discrete) proper time with the time step a_t.
- This construction is used to define the quantum amplitude in the analogy with the Feynman path integral.

Discretized Feynman path integral

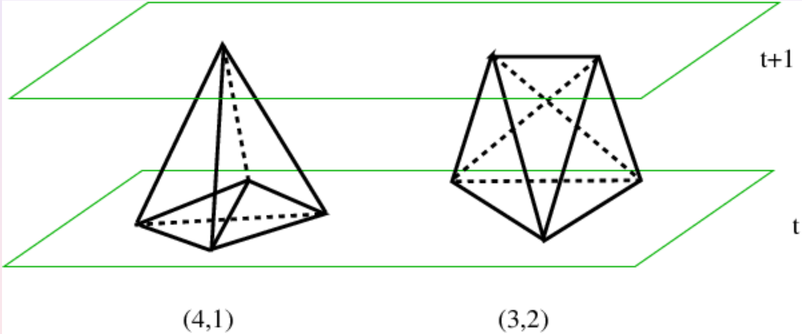
Amplitude of a transition between two geometric states

$$G(\mathbf{g}_\mathrm{i}, \mathbf{g}_\mathrm{f}, t) := \sum\!\!\!\!\!\!\!\int_{\text{geometries: } \mathbf{g}_\mathrm{i} \to \mathbf{g}_\mathrm{f}} \mathrm{e}^{iS[\mathbf{g}_{\mu\nu}(t')]}$$

Assumptions

- The spatial topology Σ is fixed in time (causality).
- The path integral can be approximated as a sum over manifolds obtained by gluing together four-dimensional simplices.
- The action can be taken as the Hilbert-Einstein action.

Building Blocks



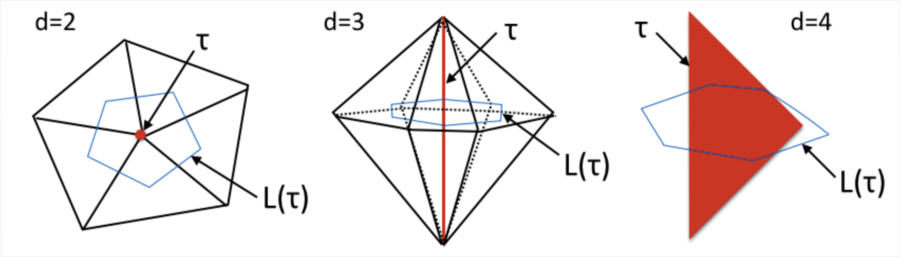
Properties

- Simplices are assumed to be internally flat.
- Simplices are connected across three-dimensional faces.
- Each simplex has five vertices, five faces (five neighboring simplices), ten edges and ten hinges (triangles).
- For each manifold we can perform the analytic continuation in a_t to obtain the Euclidean version of the amplitude (imaginary time). In this case, simplices are Euclidean simplices and each manifold is characterized by a real positive probability.

We managed to obtain a formalism without referring to any reference frame.

Consequences

- Curvature is localized in hinges. Circulating around a triangle τ (hinge) following a path L(τ) connecting centers of neighboring simplices we observe a deficit angle.



Observables

- The properties of the model in Euclidean formulation can be studied using Monte Carlo methods.
- This version corresponds to a quantum statistical model of fluctuating geometries.
- The model is completely background independent. Typical local observables are difficult to be defined.
- Global observables characterizing manifolds: averages of $N_{(4,1)}$, $N_{(3,2)}$, (numbers of simplices of a particular type), N_0 (number of vertices), T (period in time, usually we assume periodic b.c.), Hausdorff and spectral dimensions.
- More localized observables: $N_3(\tau)$ - spatial volume profile at time τ (number of $(4,1)$ simplices with the base at time τ).

Search for other observables

One of the aims of the CDT model is to relate the short-range formulation to the large-scale classical limit (hopefuly described by GR). This aim was partly successful using observables like the scale factor (existence of the de Sitter phase reproducing correctly the large-scale behavior of the scale factor and the short-scale reduction of the spectral dimension).

- The definition using deficit angles becomes singular in a naïve continuum limit.
- We would like to construct observables, which would permit the measurement and interpretation of the curvature correlations on a larger scale.
- Possibly introduce the elements of the Riemann tensor, beyond the scalar curvature, in a discretized setup.

Holonomies on a simplicial manifold

The CDT model belongs to a family of models, using Regge type simplices as building blocks.

- Following Regge, simplices are assumed to be everywhere flat on the inside.
- Geometric properties a four-simplex are completely fixed by by the edge lengths. Geometric properties of a four-geometry are encoded in the gluing data between simplices.
- The edge lengths belonging to a 3d face between two simplices must be the same.
- We do not need to introduce a coordinate system, but we are free to choose, independently for each simplex, an arbitrary Cartesian coordinate system.

Two neighboring simplices

Let us repeat the discussion about the parallel transport of vectors, assuming U_1 and U_2 to be two neighboring simplices. In this case the intersection $U_1 \cap U_2$ is the face between the simplices.

- Let us consider a path from an arbitrary point $x_1 \in U_1$ to an arbitrary point $x_2 \in U_2$, crossing the face between two simplices at an arbitrary point $\sigma \in U_1 \cap U_2$ and having otherwise an arbitrary shape.
- Since the internal geometry of the simplex is assumed to be trivial the connections Γ_1 and Γ_2 vanish and as a consequence the only contribution to the holonomy transformation comes from the rotation matrix $M^{\mu}{}_{\nu}(\sigma)$

$$M^{\mu}{}_{\nu}(\sigma) = \frac{\partial x_2^{\mu}}{\partial x_1^{\nu}}|_{\sigma} =: R(s_2, s_1)^{\mu}{}_{\nu} \in O(4)$$

.

Closed holonomy loop

This construction can be extended to define holonomy loops between any two simplices, following a path $1 \to 2 \to 3 \cdots \to n$ and a closed loop $L: \ 1 \to 2 \to 3 \cdots \to n \to 1$. The loop L is defined only by a sequence of faces (the intersections between the simplices forming a loop). The holonomy corresponding to a loop L is a product of rotation matrices

$$R_L = R(s_1, s_n) R(s_n, s_{n-1}) \cdots R(s_2, s_1)$$

The loop can be viewed as a sequence of dual links, connecting centers of simplices $1, 2 \ldots, n$. Under a change of coordinates $x_1 \to \tilde{x}_1$ the loop R_L transforms as

$$R_L \to \Lambda R_L \Lambda^T, \qquad \Lambda^\mu{}_\nu = \frac{\partial \tilde{x}_1^\mu}{\partial x_1^\nu}$$

Properties

- R_L depends only on the coordinates in the simplex "1". For a closed loop in the Euclidean setup $R_L \in SO(4)$.
- The coordinate-invariant information contained in the $SO(4)$ matrix R_L is contained in a maximal torus, parametrized by two angles θ_1 and θ_2

$$U(\theta_1, \theta_2) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

- For R_L we can define two invariants

$$t_1 = \frac{1}{2}\mathrm{Tr} R_L = \cos\theta_1 + \cos\theta_2, \quad t_2 = \frac{1}{4}\mathrm{Tr} R_L^2 + 1 = \cos^2\theta_1 + \cos^2\theta_2$$

Properties contd.

- The example of a holonomy loop is a minimal loop $L(\tau)$ around a triangle τ.
- For a minimal loop $L(\tau)$ we have a special case of a simple rotation

$$U_{L(\tau)}(\epsilon) = \begin{pmatrix} \cos\epsilon & \sin\epsilon & 0 & 0 \\ -\sin\epsilon & \cos\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  where ϵ is a sum of angles of the individual rotations around a triangle τ.
- Notice that in the CDT setting ϵ can take only one of a discrete set of values.

CDT example

The simplest case is the symmetric situation where the edges in the time and spatial directions are the same (in general we assume $a_t = \alpha\, a_s$ with $\alpha > \sqrt{7/12}$). In the symmetric case all simplices are regular simplices.

- For a simple rotation $L(\tau)$ around a triangle τ belonging to n simplices, the angle $\epsilon = n\,\phi$, where ϕ is a dihedral angle in the simplex

$$\cos\phi = \frac{1}{4}, \quad \sin\phi = \frac{\sqrt{15}}{4}$$

- We may calculate the (quantized) deficit angle. Question: What is the behavior of large holonomy loops? Is the discreteness important in this case?

Explicit construction

To calculate the holonomy for a general simplicial manifold we need to define coordinate systems in simplices forming a loop. Let us consider again a set of two neighboring simplices U and U'.

- Assume we have Cartesian coordinates $\vec{x}$ and $\vec{x}'$ in the two simplices. Each simplex has five vertices $\{1,2,3,4,5\}$ and $\{1',2',3',4',5'\}$. Let the vertices $\{1,2,3,4\}$ and $\{1',2',3',4'\}$ define the face between the simplices with an identification of $1 \leftrightarrow 1'$ etc. Vertices 5 and $5'$ are on the opposite sides of the face.
- Let the coordinates of the vertices be $\vec{x}_i,\ i = 1,\dots 5$ and $\vec{x}_i',\ i = 1,\dots 5$. These coordinates can be viewed as column matrices. Without a loss of generality we may assume that $\sum_i \vec{x}_i = \vec{0}$ and $\sum_i \vec{x}_i' = \vec{0}'$, $\vec{0} \neq \vec{0}'$.

Explicit construction cont.

- In both coordinate systems we can define the center of the face $\vec{c}_5 = \frac{1}{4}\sum_i^4 \vec{x}_i = -\frac{1}{4}\vec{x}_5$ (similarly $\vec{c}'_5 = -\frac{1}{4}\vec{x}'_5$) and the unit vector $\vec{n}_5$ (resp. $\vec{n}'_5$) orthogonal to the face. We can easily find such vectors determining the equation satisfied by the hyper-plane through the four points in both coordinate systems.
- The $O(4)$ rotation from U' to U can be obtained solving a set of four linear equations

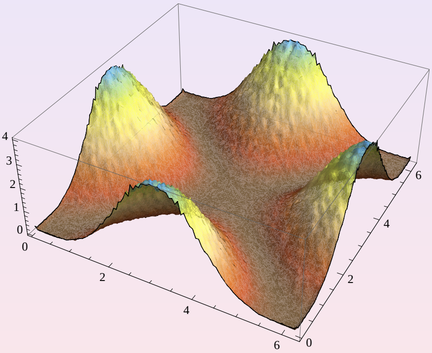
$$R \cdot (\vec{x}'_i - \vec{c}'_5 - \lambda \vec{n}'_5) = \vec{x}_i - \vec{c}_5 + \lambda \vec{n}_5, \quad i = 1, \ldots 4$$

- It can easily be checked that under this transformation $\vec{x}'_i - \vec{x}'_j \to \vec{x}_i - \vec{x}_j$ for $i, j = 1 \ldots 4$ and that $\vec{n}'_5 \to -\vec{n}_5$. Parameter $\lambda \neq 0$ is arbitrary (for $\lambda = 0$ the system of equations is singular).

Wilson lines in CDT

- We want to check statistical properties of long holonomy lines in a dynamical setup of CDT. We assume that the considered manifolds are periodic in (imaginary time) with a period $T = 80$ and with the spatial topology $\Sigma = S^3$. We fix the bare parameters of the Hilbert-Einstein action corresponding to the range in the physically most interesting de Sitter phase C.
- We consider loops in the time direction, closed by periodicity condition (corresponding to Polyakov loops in QCD on a lattice). For this particular setup the minimal length of such a loop is 320. We get a large sample of statistically independent loops with a particular length.

Distribution of θ_1 and θ_2



$$P(\theta_1, \theta_2) = \frac{1}{\pi^2} \sin^2\left(\frac{\theta_1 + \theta_2}{2}\right) \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$$

Conclusions

- For each loop we measure the two invariants t_1 and t_2 and determine the angles θ_1 and θ_2.
- The measured probability distribution reproduces perfectly the Haar measure on $SO(4)$.
- It opens a possibility to use large Wilson loops in a way similar to that in QCD.
- Large loops may be used to obtain a macroscopic information about curvature correlations.
- Work in progress on measuring this type of observables in other phases of the model and for different topologies Σ.
- New project: Introducing the coupling between the CDT dynamical geometry and a massive (or massless) vector field conformally coupled using the formalism proposed above.

Thank you
and

Happy Birthday !!