Geometry of null hypersurfaces

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Abstract:
We discuss geometry of null surfaces (and its possible applications to the horizons, null shells, near horizon geometry, thermodynamics of black holes)

Old ideas

In Synge's festshrift volume [GR, O'Raifeartaigh, Oxford 1972, 101-15] Roger Penrose distinguished three basic structures which a null hypersurface N in four-dimensional spacetime M acquires from the ambient Lorentzian geometry:

- the degenerate metric $g|_N$ (see [P. Nurowski, D.C Robinson, CQG 17 (2000) 4065-84] for Cartan's classification of them and the solution of the local equivalence problem)
- the concept of an affine parameter along each of the null geodesics from the two-parameter family ruling N
- the concept of parallel transport for tangent vectors along each of the null geodesics

Geometric structures on screen distribution

Natural geometric structures on TN/K – *screen distribution*

- time-oriented Lorentzian manifold M $(-,+,+,+)$
- null hypersurface N – submanifold with codim=1 with degenerate induced metric $g|_N$ $(0,+,+)$, K – time-oriented non-vanishing null vector field such that $K_p^\perp = T_pN$ at each point $p \in N$
  1. K is null and tangent to N, $g(X,K)$=0 iff $X \in \Gamma TN$
  2. integral curves of K – null geodesic generators of N
  3. K is determined by N up to a scaling factor – positive function

Screen distribution TN/K

- $T_pN/K := \{\overline{X} : X \in T_pN\}$ where $\overline{X} = [X]_{\mathrm{mod}\,K}$ is an equivalence class of the relation $\mathrm{mod}\,K$ defined as follows:

$$X \equiv Y (\mathrm{mod}\,K) \iff X - Y \text{ is parallel to } K$$

- $TN/K := \cup_{p\in N} T_pN/K$ vector bundle over N with 2-dimensional fibers (equipped with Riemannian metric h), the structure does not depend on the choice of K (scaling factor)

$$h : T_pN/K \times T_pN/K \longrightarrow \mathbb{R}, \quad h(\overline{X}, \overline{Y}) = g(X, Y)$$

Remark: If $t(K, \cdot) = 0$ then $\bar{t}(\bar{X}, \cdot)$ can be correctly defined on TN/K. This implies that g, b, B are well defined on TN/K.

Null Weingarten map and second fundamental form

- null Weingarten map $\bar{b}_K$ (depending on the choice of scaling factor, in non-degenerate case one can always take unit normal to the hypersurface but in null case the vectorfield K is no longer transversal to N and has always scaling factor freedom because its length vanishes)

$$\bar{b}_K : T_pN/K \longrightarrow T_pN/K\,, \quad \bar{b}_K(\overline{X}) = \overline{\nabla_X K}$$

$$\bar{b}_{fK} = f\,\bar{b}_K\,, \quad f \in C^\infty(N)\,, \quad f > 0$$

- null second fundamental form $\bar{B}_K$ (bilinear form associated to $\bar{b}_K$ via h)

$$\bar{B}_K : T_pN/K \times T_pN/K \longrightarrow \mathbb{R}$$

$$\bar{B}_K(\overline{X},\overline{Y}) = h(\bar{b}_K(\overline{X}),\overline{Y}) = g(\nabla_X K, Y)$$

$\bar{b}_K$ is self-adjoint with respect to h and $\bar{B}_K$ is symmetric

Null mean curvature

- N is totally geodesic (i.e. restriction to N of the Levi-Civita connection of M is an affine connection on N, any geodesic in M starting tangent to N stays in N) $\iff B = 0$,
  (*non-expanding horizon is a typical example*)
  $0 = (\nabla_X K | Y) \Rightarrow \nabla_X K = w(X) K$
- null mean curvature of N with respect to K

$$\theta := \operatorname{tr} b = \sum_{i=1}^{2} \bar{B}_K(\overline{e}_i, \overline{e}_i) = \sum_{i=1}^{2} g(\nabla_{e_i} K, e_i)$$

S – two-dimensional submanifold of N transverse to K,

e_i – orthonormal basis for $T_p S$ in the induced metric,
$\overline{e}_i$ – orthonormal basis for $T_p N/K$

Curvature endomorphism, Raychaudhuri equation

Assume that K is an (affine-)geodesic vector field i.e. $\boxed{\nabla_K K = 0}$

We denote by $'$ covariant differentiation in the null direction:

$$\overline{Y}' := \overline{\nabla_K Y}\,, \quad b'(\overline{Y}) := b(\overline{Y})' - b(\overline{Y}')$$

curvature endomorphism

$$R : T_p N/K \longrightarrow T_p N/K\,, \quad R(\overline{X}) = \overline{Riemann(X,K)K}$$

Ricatti equation

$$b' + b^2 + R = 0$$

Taking the trace we obtain well-known *Raychaudhuri equation*:

$$\theta' = -Ricci(K,K) - B^2\,, \quad B^2 = \sigma^2 + \frac{1}{2}\theta^2 \qquad (1)$$

σ – shear scalar corresponding to the trace free part of B

Well-known fact

The following proposition is a standard application of the Raychaudhuri equation.

Proposition

Let M be a spacetime which obeys the null energy condition i.e. $Ricci(X,X) \geq 0$ for all null vectors X, and let N be a smooth null hypersurface in M. If the null generators of N are future geodesically complete then N has non-negative null mean curvature i.e. $\theta \geq 0$.

Weingarten map – two possibilities

$$b(X) = \nabla_X K\,, \quad \bar{b}(\bar{X}) = \overline{\nabla_X K}\,, \quad \pi(X) = \bar{X}$$

$$\begin{array}{ccc}
TN & \xrightarrow{\pi} & TN/K \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle \bar{b}} \\
TN & \xrightarrow{\pi} & TN/K \\
& \searrow \quad \swarrow & \\
& N &
\end{array}$$

Properties of b and B on TN

Properties of $\nabla K : TN \to TN$ and $g(\nabla K) : TN \to T^*N$

b and B

$$b(X) := \nabla_X K\,, \quad B(X,Y) := g(\nabla_X K, Y)$$

- $\nabla_X (K|K) = 0 \Rightarrow (\nabla_X K|K) = 0 \Rightarrow b(TN) \subset TN$
- B is symmetric and bilinear
- $\pounds_K g = 2B$ ($\pounds$ – Lie derivative)
- K is geodesic
- for all X $(b(X)|K) = 0 \Rightarrow B(\cdot, K) = 0$
- $B_{fK}(X,Y) = fB_K(X,Y)$ (scaling)

Questions:

- What is the analog of canonical ADM momentum for the null surface?
- What are the "initial value constraints"?
- Are they intrinsic objects?

Applications:

- Dynamics of the light-like matter shell from matter Lagrangian which is an invariant scalar density on N [*Dynamics of a self gravitating light-like matter shell: a gauge-invariant Lagrangian and Hamiltonian description*, Physical Review D **65** (2002), 064036]
- Dynamics of gravitational field in a finite volume with null boundary and its application to black holes thermodynamics [*Dynamics of gravitational field within a wave front and thermodynamics of black holes*, Physical Review D **70** (2004), 124010]

Non-degenerate hypersurface – reminder

Canonical ADM momentum

$$P^{kl} = \sqrt{\det g_{mn}}(g^{kl} TrK - K^{kl})$$

where K^{kl} is the second fundamental form (external curvature) of the embedding of the hypersurface into the space-time M

Gauss-Codazzi equations for non-degenerate hypersurface

$$P_i{}^l{}_{|l} = \sqrt{\det g_{mn}}\, G_{i\mu} n^\mu \quad (= 8\pi \sqrt{\det g_{mn}}\, T_{i\mu} n^\mu)$$

$$(\det g_{mn})\mathcal{R} - P^{kl} P_{kl} + \frac{1}{2}(P^{kl} g_{kl})^2 = 2(\det g_{mn})\, G_{\mu\nu} n^\mu n^\nu$$

$$(= 16\pi (\det g_{mn})\, T_{\mu\nu} n^\mu n^\nu)$$

$\mathcal{R}$ is the (three–dimensional) scalar curvature of g_{kl},
n^μ is a four–vector normal to the hypersurface,
$T_{\mu\nu}$ is an energy–momentum tensor of the matter field,
and the calculations have been made with respect to the non-degenerate induced three–metric g_{kl} ("|" denotes covariant derivative, indices are raised and lowered etc.)

Coordinates convention

A null hypersurface in a Lorentzian spacetime M is a three-dimensional submanifold $N \subset M$ such that the restriction g_{ab} of the spacetime metric $g_{\mu\nu}$ to N is degenerate.
We shall often use adapted coordinates:

- coordinate x^3 is constant on N.
- Space coordinates will be labeled by $k, l = 1, 2, 3$;
- coordinates on N will be labeled by $a, b = 0, 1, 2$;
- coordinates on S will be labeled by $A, B = 1, 2$.
- Spacetime coordinates will be labeled by Greek characters α, β, μ, ν.

Repetition in coordinates

The non-degeneracy of the spacetime metric implies that the metric g_{ab} induced on N from the spacetime metric $g_{\mu\nu}$ has signature $(0,+,+)$. This means that there is a non-vanishing null-like vector field K^a on N, such that its four-dimensional embedding K^μ to M (in adapted coordinates $K^3=0$) is orthogonal to N. Hence, the covector $K_\nu = K^\mu g_{\mu\nu} = K^a g_{a\nu}$ vanishes on vectors tangent to N and, therefore, the following identity holds:

$$K^a g_{ab} \equiv 0 \ . \tag{2}$$

It is easy to prove that integral curves of K^a are geodesic curves of the spacetime metric $g_{\mu\nu}$. Moreover, any null hypersurface N may always be embedded in a one-parameter congruence of null hypersurfaces.

Volume element

Since our considerations are purely local, we fix the orientation of the $\mathbb{R}^1$ component (along K) in $N = \mathbb{R}^1 \times S^2$ and assume that null-like vectors K describing degeneracy of the metric g_{ab} of N will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate x^0 increases in the direction of K, i.e., inequality $K(x^0) = K^0 > 0$ holds. In these coordinates degeneracy fields are of the form $K = f(\partial_0 - n^A \partial_A)$, where $f > 0$, $n_A = g_{0A}$ and we rise indices with the help of the two-dimensional matrix $\tilde{\tilde{g}}^{AB}$, inverse to g_{AB}.
Denote by λ the two-dimensional volume form on each surface $x^0 = \mathrm{const}$:

$$\lambda := \sqrt{\det g_{AB}} \ , \tag{3}$$

then for any degeneracy field K of g_{ab} the following object

$$v_K := \frac{\lambda}{K(x^0)}$$

is a well defined scalar density on N. This means that

$$\mathbf{v}_K := v_K dx^0 \wedge dx^1 \wedge dx^2$$

is a coordinate-independent differential three-form on N. However, v_K depends upon the choice of the field K.

It follows immediately from the above definition that the following object:

$$\Lambda = v_K \, K$$

is a well defined (i.e., coordinate-independent) vector density on N. Obviously, it *does not depend* upon any choice of the field K:

$$\Lambda = \lambda(\partial_0 - n^A \partial_A) \tag{4}$$

and it is an intrinsic property of the internal geometry g_{ab} of N. The same is true for the divergence $\partial_a \Lambda^a$, which is, therefore, an invariant, K-independent, scalar density on N. Mathematically (in terms of differential forms), the quantity Λ represents the two-form:

$$\mathbf{L} := \Lambda^a \left(\partial_a \rfloor dx^0 \wedge dx^1 \wedge dx^2 \right) \;,$$

whereas the divergence represents its exterior derivative (a three-from): $\mathrm{d}\mathbf{L} := (\partial_a \Lambda^a) \, dx^0 \wedge dx^1 \wedge dx^2$.
In particular, a null surface with vanishing $\mathrm{d}\mathbf{L}$ is the *non-expanding horizon*.

$\mathbf{L}$ and $\mathbf{v}_K$ without coordinates

Both objects $\mathbf{L}$ and $\mathbf{v}_K$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $p \in N$, the tangent space T_pN may be quotiented with respect to the degeneracy subspace spanned by K. The quotient space T_pN/K carries a non-degenerate Riemannian metric h and, therefore, is equipped with a volume form ω (its coordinate expression would be: $\omega = \lambda \, dx^1 \wedge dx^2$). The two-form $\mathbf{L}$ is equal to the pull-back of ω from the quotient space T_pN/K to T_pN.

$$\pi : T_pN \longrightarrow T_pN/K \, , \quad \mathbf{L} := \pi^*\omega$$

The three-form $\mathbf{v}_K$ may be defined as a product:

$$\mathbf{v}_K = \alpha \wedge \mathbf{L} \, ,$$

where α is *any* one-form on N, such that $< K, \alpha > \equiv 1$.
We have

$$\mathrm{d}\mathbf{L} = \theta \mathbf{v}_K$$

where θ is a null mean curvature of N.

Connection needs (null) isometry

The degenerate metric g_{ab} on N does not allow to define *via* the compatibility condition $\nabla g = 0$, any natural connection, which could be applied to generic tensor fields on N.
Moreover, such connection drastically reduces the degenerate metric structure g on N.
Existence of any symmetric connection $\bar{\nabla}$ on N compatible with g implies $\pounds_K g = 0$ and N becomes totally geodesic.

symmetric connection implies trivial second fundamental form

$$\bar{\nabla} g = 0 \Rightarrow \pounds_K g = 0$$

Divergence of (contravariant-covariant) tensor density

Nevertheless, there is one exception: the degenerate metric defines *uniquely* a certain covariant, first order differential operator. The operator may be applied only to mixed (contravariant-covariant) tensor density fields $\mathbf{H}^a{}_b$, satisfying the following algebraic identities:

algebraic properties of H needed for divergence

$$\mathbf{H}^a{}_b K^b = 0 \ , \tag{5}$$

$$\mathbf{H}_{ab} = \mathbf{H}_{ba} \ , \tag{6}$$

where $\mathbf{H}_{ab} := g_{ac} \mathbf{H}^c{}_b$. Its definition cannot be extended to other tensorial fields on N. Fortunately, the extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor densities of this type.

The operator, which we denote by $\overline{\nabla}_a$, is defined by means of the four-dimensional metric connection in the ambient spacetime M in the following way:
Given $\mathbf{H}^a{}_b$, take any its extension $\mathbf{H}^{\mu\nu}$ to a four-dimensional, symmetric tensor density, "orthogonal" to N, i.e. satisfying $\mathbf{H}^{\perp\nu} = 0$ ("$\perp$" denotes the component transversal to N). Define $\overline{\nabla}_a \mathbf{H}^a{}_b$ as the restriction to N of the four-dimensional covariant divergence $\nabla_\mu \mathbf{H}^\mu{}_\nu$.

Divergence of tensor density

The ambiguities which arise when extending three-dimensional object $\mathbf{H}^a{}_b$ living on N to the four-dimensional one cancel out and the result is unambiguously defined as a covector density on N. It turns out, however, that this result does not depend upon the spacetime geometry and may be defined intrinsically on N as follows:

$$\nabla_a \mathbf{H}^a{}_b \;\; = \;\; \partial_a \mathbf{H}^a{}_b - \frac{1}{2} \mathbf{H}^{ac} g_{ac,b} \; ,$$

where $g_{ac,b} := \partial_b g_{ac}$, a tensor density $\mathbf{H}^a{}_b$ satisfies identities (5) and (6), and moreover, $\mathbf{H}^{ac}$ is *any* symmetric tensor density, which reproduces $\mathbf{H}^a{}_b$ when lowering an index:

$$\mathbf{H}^a{}_b = \mathbf{H}^{ac} g_{cb} \, . \tag{7}$$

It is easily seen, that such a tensor density always exists due to identities (5) and (6), but the reconstruction of $\mathbf{H}^{ac}$ from $\mathbf{H}^a{}_b$ is not unique, because $\mathbf{H}^{ac} + CK^a K^c$ also satisfies (7) if $\mathbf{H}^{ac}$ does. Conversely, two such symmetric tensors $\mathbf{H}^{ac}$ satisfying (7) may differ only by $CK^a K^c$. Fortunately, this non-uniqueness does not influence the value of (7).

Intrinsic divergence of tensor density

Hence, the following definition makes sense:

intrinsic divergence on N

$$\overline{\nabla}_a \mathbf{H}^a{}_b := \partial_a \mathbf{H}^a{}_b - \frac{1}{2} \mathbf{H}^{ac} g_{ac,b} \ . \tag{8}$$

The right-hand-side does not depend upon any choice of coordinates (i.e., transforms like a genuine covector density under change of coordinates).

To express the result directly in terms of the original tensor density $\mathbf{H}^a{}_b$, we observe that it has five independent components and may be uniquely reconstructed from $\mathbf{H}^0{}_A$ (2 independent components) and the symmetric two-dimensional matrix $\mathbf{H}_{AB}$ (3 independent components). Indeed, identities (5) and (6) may be rewritten as follows:

$$\mathbf{H}^A_B = \tilde{\tilde{g}}^{AC}\mathbf{H}_{CB} - n^A\mathbf{H}^0{}_B \ , \tag{9}$$

$$\mathbf{H}^0_0 = \mathbf{H}^0{}_A n^A \ , \tag{10}$$

$$\mathbf{H}^B_0 = \left(\tilde{\tilde{g}}^{BC}\mathbf{H}_{CA} - n^B\mathbf{H}^0{}_A\right) n^A \ . \tag{11}$$

The correspondence between $\mathbf{H}^a{}_b$ and $(\mathbf{H}^0{}_A, \mathbf{H}_{AB})$ is one-to-one.

To reconstruct $\mathbf{H}^{ab}$ from $\mathbf{H}^a{}_b$ up to an arbitrary additive term CK^aK^b, take the following, coordinate dependent, symmetric quantity:

$$\mathbf{F}^{AB} := \tilde{\tilde{g}}^{AC}\mathbf{H}_{CD}\tilde{\tilde{g}}^{DB} - n^A\mathbf{H}^0{}_C\tilde{\tilde{g}}^{CB} - n^B\mathbf{H}^0{}_C\tilde{\tilde{g}}^{CA} \, , \tag{12}$$

$$\mathbf{F}^{0A} := \mathbf{H}^0{}_C\tilde{\tilde{g}}^{CA} =: \mathbf{F}^{A0} \, , \tag{13}$$

$$\mathbf{F}^{00} := 0 \, . \tag{14}$$

It is easy to observe that any $\mathbf{H}^{ab}$ satisfying (7) must be of the form:

$$\mathbf{H}^{ab} = \mathbf{F}^{ab} + \mathbf{H}^{00}K^aK^b \, . \tag{15}$$

The non-uniqueness in the reconstruction of $\mathbf{H}^{ab}$ is, therefore, completely described by the arbitrariness in the choice of the value of $\mathbf{H}^{00}$. Using these results, we finally obtain:

$$\begin{aligned} \overline{\nabla}_a\mathbf{H}^a{}_b \quad &:= \quad \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\mathbf{H}^{ac}g_{ac,b} = \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\mathbf{F}^{ac}g_{ac,b} \\ &= \quad \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\left(2\mathbf{H}^0{}_A \; n^A{}_{,b} - \mathbf{H}_{AC}\tilde{\tilde{g}}^{AC}{}_{,b}\right) . \end{aligned} \tag{16}$$

The operator on the right-hand-side of (16) is called the (three-dimensional) covariant derivative of $\mathbf{H}^a{}_b$ on N with respect to its degenerate metric g_{ab}. It is well defined (i.e., coordinate-independent) for a tensor density $\mathbf{H}^a{}_b$ fulfilling conditions (5) and (6). One can also show that the above definition coincides with the one given in terms of the four-dimensional metric connection and due to (7), it equals:

divergence on N induced from ambient M

$$\nabla_\mu \mathbf{H}^\mu{}_b = \partial_\mu \mathbf{H}^\mu{}_b - \frac{1}{2}\mathbf{H}^{\mu\lambda} g_{\mu\lambda,b} = \partial_a \mathbf{H}^a{}_b - \frac{1}{2}\mathbf{H}^{ac} g_{ac,b} \ , \qquad (17)$$

and, whence, coincides with $\overline{\nabla}_a \mathbf{H}^a{}_b$ defined intrinsically on N.

Canonical tensor density – analog of ADM momentum

To describe exterior geometry of N we begin with covariant derivatives *along* N of the "orthogonal vector K". Consider the tensor $\nabla_a K^\mu$. Unlike the non-degenerate case, there is no unique "normalization" of K and, therefore, such an object does depend upon a choice of the field K. The length of K vanishes. Hence, the tensor is again orthogonal to N, i.e., the components corresponding to $\mu = 3$ vanish identically in adapted coordinates. This means that $\nabla_a K^b$ is a purely three-dimensional tensor living on N. For our purposes it is useful to use the "ADM-momentum" version of this object, defined in the following way:

null "ADM-momentum"

$$Q^a{}_b(K) := -s\left\{ v_K \left(\nabla_b K^a - \delta^a_b \nabla_c K^c \right) + \delta^a_b \partial_c \Lambda^c \right\}, \qquad (18)$$

where $s := \operatorname{sgn} g^{03} = \pm 1$. Due to above convention, the object $Q^a{}_b(K)$ feels only *external orientation* of N and does not feel any internal orientation of the field K.

Remark: If N is a *non-expanding horizon*, the last term in the above definition vanishes.

Comment to the definition of Q

The last term $\delta^a_b \nabla_c K^c$ in (18) is K-independent. It has been introduced in order to correct algebraic properties of the quantity

$$v_K \left(\nabla_b K^a - \delta^a_b \nabla_c K^c \right) .$$

One can show that ${Q^a}_b$ satisfies identities (5)–(6) and, therefore, its covariant divergence with respect to the degenerate metric g_{ab} on N is uniquely defined.

Div of Q versus Einstein tensor density

This divergence enters into the Gauss–Codazzi equations, which relate the divergence of Q with the transversal component $\mathcal{G}^{\perp}{}_{b}$ of the Einstein tensor density

$$\mathcal{G}^{\mu}{}_{\nu} = \sqrt{|\det g|} \left(R^{\mu}{}_{\nu} - \frac{1}{2} \delta^{\mu}{}_{\nu} R \right) .$$

The transversal component of such a tensor density is a well defined three-dimensional object living on N. In coordinate system adapted to N, i.e., such that the coordinate x^3 is constant on N, we have $\mathcal{G}^{\perp}{}_{b} = \mathcal{G}^{3}{}_{b}$. Due to the fact that $\mathcal{G}$ is a tensor density, components $\mathcal{G}^{3}{}_{b}$ *do not change* with changes of the coordinate x^3, provided it remains constant on N. These components describe, therefore, an intrinsic covector density living on N.

Vector constraint on N

Proposition

The following null-like-surface version of the Gauss–Codazzi equation is true:

$$\overline{\nabla}_a Q^a{}_b(K) + s v_K \partial_b \left(\frac{\partial_c \Lambda^c}{v_K} \right) \equiv -\mathcal{G}^{\perp}{}_b \ . \qquad (19)$$

We remind that the ratio between two scalar densities: $\partial_c \Lambda^c$ and v_K, is a scalar function θ. Its gradient is a covector field. Finally, multiplied by the density v_K, it produces an intrinsic covector density on N. This proves that also the left-hand-side is a well defined geometric object living on N.
The component $K^b \mathcal{G}^{\perp}{}_b$ of the equation (19) is nothing but a densitized form of Raychaudhuri equation (1) for the congruence of null geodesics generated by the vector field K.

2+1 decomposition of constraints

$$K = \partial_0 - n^A \partial_A\,, \quad \overline{\nabla}_a Q^a{}_b(K) + s v_K \partial_b \left(\frac{\partial_c \Lambda^c}{v_K} \right) \equiv -\mathcal{G}^\perp{}_b$$

The quantities $l_{AB} := -B_{AB} = -g_{Ac} b^c{}_B$ and $w_a = -b^0{}_a$ represent 2+1 decomposition of Weingarten map $b^a{}_b$.

$$K^a \partial_a l + (w_a K^a)\, l - \frac{1}{2} l^2 - \bar{l}^{AB} \bar{l}_{AB} = 8\pi T_{ab} K^a K^b \ ,$$

where we have decomposed l_{AB} into its trace l (expansion) and its traceless part $\bar{l}_{AB} := l_{AB} - \frac{1}{2} g_{AB} l$ (shear).

$w_a K^a$ corresponds to *surface gravity*.

$$\partial_0 w_B - w_{B\|A} n^A - w_A n^A_{\|B} - (w_a K^a)_{\|B} - w_B l + \bar{l}^A_{B\|A} - \frac{1}{2} l_{\|B} = -8\pi T_{aB} K^a$$

In case of vacuum spacetimes the right-hand sides of the above constraints vanish.

(Foliation dependent) fourth constraint

if we add slicing of N as an additional structure we can derive fourth constraint:

"half"-intrinsic on N fourth constraint

$$-G(K,Z) = (\partial_0 - w_0 - l)\,k + \frac{1}{2}\overset{(2)}{R} + w^A{}_{||A} - w_A w^A$$

where $\overset{(2)}{R}$ is a scalar curvature of the Riemannian metric structure g_{AB} and Z is null

$$g(K,K) = g(Z,Z) = 0\,, \quad g(K,Z) = 1\,, \quad g(K,\partial_A) = g(Z,\partial_A) = 0$$

$$l_{ab} = K^\mu \Gamma_{\mu ab}\,, \quad k_{ab} = Z^\mu \Gamma_{\mu ab}\,,$$

$$-w_a = Z^\mu K^\nu \Gamma_{\mu\nu a} = K^b k_{ab}\,, \quad K^b l_{ab} = 0$$

Conclusions/Applications

- Crossing null shells – Dray-t'Hooft-Redmount formula
- B=0 (totally geodesic null surface) – (non-expanding, Killing) horizons, Near Horizon Geometry
- Constraints – three of them are intrinsic and correspond to divergence of tensor density, the fourth one needs extra structure – foliation of N
- Q and g play a role of 'initial/boundary data' on N, they can be used to define *local* first law of black hole thermodynamics for privileged field K
- Q and g reduce to covector w_A and two-metric g_{AB} in the case of Near Horizon Geometry and vacuum Einstein equations lead to *basic equation*:

$$w^A{}_{||B} + w_B{}^{||A} + 2w^A w_B = R^A{}_B = \frac{1}{2} R \delta^A{}_B \,, \qquad (20)$$

where w^A is a vector field, $w_B = g_{AB} w^B$, $||$ denotes covariant derivative with respect to the metric g_{AB} and R_{AB} is its Ricci tensor.

- The above equation appears not only in the context of Kundt's class, it also arises in the study of vacuum degenerate isolated horizons.
- Any degenerate Killing horizon also implies this equation.
- For axial symmetry and spherical topology there is a unique solution – extremal Kerr.
- When one-form $w_B \mathrm{d}x^B$ is closed (e.g. static degenerate horizon) there are no solutions of (20). However, in general, the space of solutions is not known.

Linearized basic equation

Existence of (general non-symmetric) solutions to linearized *basic equation* around Kerr has the answer – no solutions

J. Jezierski, B. Kamiński: *Towards uniqueness of degenerate axially symmetric Killing horizon*, Gen Relativ Gravit **45** (2013) 987-1004, DOI 10.1007/s10714-013-1506-0, arXiv: 1206.5136 [gr-qc]
PT Chruściel, SJ Szybka and P Tod: *Towards a classification of vacuum near-horizons geometries*, Class. Quantum Grav. **35** (2018) 015002

Green function on a sphere $\longrightarrow$ extremal Kerr

Extremal Kerr has natural representation (in NHG) by generalized Green function

J. Jezierski: *On the existence of Kundt's metrics and degenerate (or extremal) Killing horizons*, Class. Quantum Grav. **26** (2009) 035011

Solution of the problem with axial symmetry

$$g = 2m^2 \left[\frac{1+\cos^2\theta}{2} d\theta^2 + \frac{2\sin^2\theta}{1+\cos^2\theta} d\varphi^2 \right] \tag{21}$$

$$w^\theta = -\frac{\sin\theta\cos\theta}{m^2(1+\cos^2\theta)^2}, \quad w^\varphi = \frac{1}{2m^2(1+\cos^2\theta)}, \tag{22}$$

represents extremal Kerr with mass m and angular momentum m^2.

Solution of the problem with axial symmetry

It is worth to notice that the Kerr solution (22) in terms of

$$\Phi_A := \frac{w_A}{w^B w_B}$$

has a simple and natural form.
Linear equations for Φ_A extended through the "poles" are

linear part of basic equation

$$\Phi_{A||C}\varepsilon^{AC} = 4\pi m^2 \left(\delta_{\theta=\pi} - \delta_{\theta=0}\right) , \tag{23}$$

$$\Phi^A{}_{||A} = 1 - 4\pi m^2 \left(\delta_{\theta=\pi} + \delta_{\theta=0}\right) , \tag{24}$$

where by δ_p we denote a Dirac delta at point p and $8\pi m^2 (= \int \lambda)$ is a total volume of the (Kerr) sphere (21).

Solution of the problem with axial symmetry

Let G_p be a Green function satisfying

$$\begin{cases} \triangle G_p = 1 - 8\pi m^2 \delta_p \,, \\ \int \lambda G_p = 0 \,. \end{cases} \tag{25}$$

The potentials Φ, $\tilde{\Phi}$ for the covector field Φ_A defined (up to a constant) as follows

$$\Phi_A = \partial_A \Phi + {\varepsilon_A}^B \partial_B \tilde{\Phi} \tag{26}$$

take a simple form

$$\begin{aligned} \Phi &= \frac{1}{2}(G_{\theta=0} + G_{\theta=\pi}) \\ \tilde{\Phi} &= \frac{1}{2}(G_{\theta=0} - G_{\theta=\pi}) \end{aligned}$$

because equations (23), (24) and (26) imply

$$\begin{aligned} \triangle\Phi &= 1 - 4\pi m^2 \left(\delta_{\theta=\pi} + \delta_{\theta=0}\right) , \\ \triangle\tilde{\Phi} &= 4\pi m^2 \left(\delta_{\theta=\pi} - \delta_{\theta=0}\right) . \end{aligned}$$

Green functions for extremal Kerr (21)

closed form of Green function for extremal Kerr

$$\begin{aligned} G_{\theta=0} &= 4m^2 \left[\frac{1}{2}\sin^2\frac{\theta}{2} + \frac{1}{8}\sin^2\theta - \log(\sin\frac{\theta}{2}) + \frac{1}{3}\right] , \\ G_{\theta=\pi} &= 4m^2 \left[\frac{1}{2}\cos^2\frac{\theta}{2} + \frac{1}{8}\sin^2\theta - \log(\cos\frac{\theta}{2}) + \frac{1}{3}\right] . \end{aligned}$$