Uniqueness

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1 Canonical Quantization

- Given: classical system with first-class constraints

1. Elementary Variables
   - choose separating space $\mathfrak{S}$ of phase space functions
2. Quantization
   - choose "representation" of $\mathfrak{S}$ on some kinematical Hilbert space $\mathcal{H}$, giving self-adjoint constraints
3. Group Averaging
   - choose constraint-invariant dense subset Φ in Hilbert space $\mathcal{H}$
   - solve constraints using Gelfand triple $\Phi \subseteq \mathcal{H} \subseteq \Phi'$

$$\eta(\phi) \quad := \quad \int_{\mathcal{Z}} \mathrm{d}\mu(Z)\ \overline{Z\phi} \ \in \ \Phi'$$

4. Physical Hilbert Space
   - inner product: $\langle \eta\phi_1, \eta\phi_2 \rangle_{\mathsf{phys}} \ := \ (\eta\phi_1)[\phi_2]$
   - completion of $\eta(\Phi)$ gives physical Hilbert space, self-adjoint dual representation of observable algebra

Ashtekar, Lewandowski, Marolf, Mourão, Thiemann 1995

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thieMann, mourAo, marolF, lewandowskI, ashtekAr 1995

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2 Basics

- Given: Ashtekar gravity (A, E)

1. Elementary Variables
    - choose separating space $\mathfrak{S}$ of phase space functions

- Basic functions

$$h_\gamma(A) := \mathcal{P}e^{\int_\gamma A}$$

$$E_{S,f} := \int_S [*E](f)$$

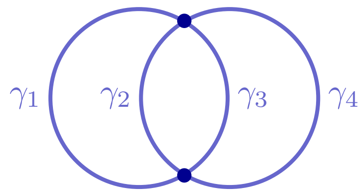
- Cylindrical functions

$$\psi := \psi_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}$$

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$

$\psi_{\boldsymbol{\gamma}}$: smooth

$\pi_{\boldsymbol{\gamma}} = h_{\gamma_1} \times \cdots \times h_{\gamma_n} : \mathcal{A} \longrightarrow \mathbf{G}^n$



- Derivations on Cyl

$$X_{S,f}\psi := \{\psi, E_{S,f}\}$$

- Diffeos act via $\boldsymbol{\gamma}$, S, f, e.g.:

$$\alpha_\Psi(f \circ h_{\boldsymbol{\gamma}}) := f \circ h_{\Psi(\boldsymbol{\gamma})}$$

- Holonomy-Flux Algebra

$\mathfrak{H}$ ... $*$-algebra of all words in Cyl and $\mathfrak{X}$ factorized by the relations

$$\begin{aligned} a \cdot X - X \cdot a &= \mathrm{i}\,\{a, X\} && (\text{CCR}, a \in \mathrm{Cyl} \cup \mathfrak{X}) \\ \psi \cdot \psi' &= \psi\,\psi' && (\text{Cyl-module}) \end{aligned} \quad + \text{ linearity}$$

- Standard Invariant State ω_0

$$\begin{aligned} \omega_0(a \cdot X) &= 0 && (a \in \mathfrak{H}, X \in \mathfrak{X}) \\ \omega_0(\psi) &= \int_{\mathbf{G}^n} \psi_\gamma \,\mathrm{d}\mu_{\text{Haar}} && (\psi = \psi_\gamma \circ \pi_\gamma \in \mathrm{Cyl}) \end{aligned}$$

Theorem: Lewandowski, Okołów, Sahlmann, Thiemann 2005

Assume
- $\dim M \geq 2$
- hypersurfaces – semianalytic
- diffeos – semianalytic
- smearings with compact support

Then ω_0 is the **unique** state on $\mathfrak{H}$ that is invariant w.r.t. bundle automorphisms.

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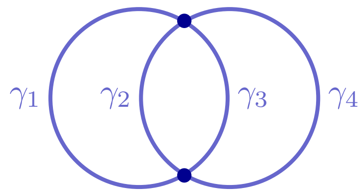
$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$; $\psi_{\boldsymbol{\gamma}}$: smooth; $\pi_{\boldsymbol{\gamma}} = h_{\gamma_1} \times \cdots \times h_{\gamma_n} : \mathcal{A} \longrightarrow \mathbf{G}^n$



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- Given: Ashtekar gravity (A, E) + homogeneity + isotropy

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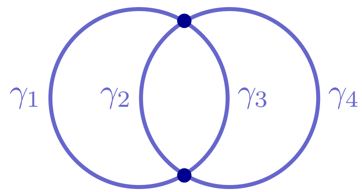

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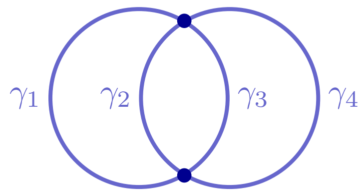

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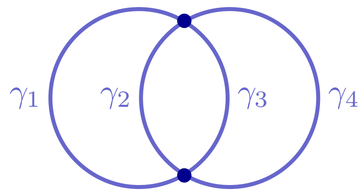
restricted to $\mathbb{R}$

dense in $\mathfrak{A} := C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$



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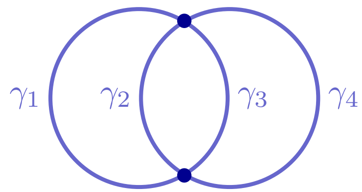

- Derivations on Cyl

$$X_{S,f}\psi := \{\psi, E_{S,f}\}$$

$X = \frac{\mathrm{d}}{\mathrm{d}t}$

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   - choose separating space $\mathfrak{S}$ of phase space functions

- Basic functions
$$h_\gamma(A) \overset{\text{restricted}}{:=} \mathcal{P}e^{\int_\gamma A}$$
$$E_{S,f} \overset{\text{not smeared}}{:=} \int_S [*E](f)$$

- Cylindrical functions

$$\psi := \psi_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}$$

$\psi_{\boldsymbol{\gamma}}$: smooth; restricted to $\mathbb{R}$; dense in $\mathfrak{A} := C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$

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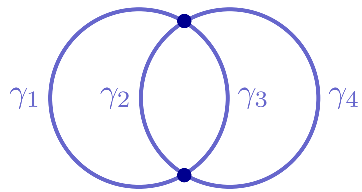

- Derivations on Cyl
$$X_{S,f}\psi := \{\psi, E_{S,f}\} \qquad X = \frac{\mathrm{d}}{\mathrm{d}t}$$

- Diffeos (residual dilations) act via $\boldsymbol{\gamma}$, S, f, e.g.:
$$\alpha_\Psi(f \circ h_{\boldsymbol{\gamma}}) := f \circ h_{\Psi(\boldsymbol{\gamma})}$$

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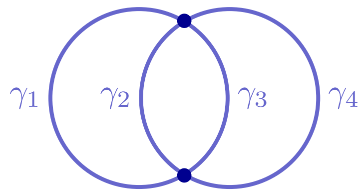

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$$X_{S,f}\psi := \{\psi, E_{S,f}\}$$

$X = \frac{\mathrm{d}}{\mathrm{d}t}$

- Diffeos act via $\boldsymbol{\gamma}$, S, f, e.g.: residual dilations

$$\alpha_\Psi(f \circ h_{\boldsymbol{\gamma}}) := f \circ h_{\Psi(\boldsymbol{\gamma})}$$

$\alpha_\lambda(X) = \lambda X$

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Theorem: Lewandowski, Okołów, Sahlmann, Thiemann 2005

Assume
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Then ω_0 is the **unique** state on $\mathfrak{H}$ that is invariant w.r.t. bundle automorphisms.

- Restricted Holonomy-Flux Algebra

$\mathfrak{H}_{\mathrm{C}}$ ... $*$-algebra of all words in $\mathfrak{B}$ and $\mathfrak{X}$ factorized by the relations

$$\begin{aligned} a \cdot X - X \cdot a &= \mathrm{i}\,\{a, X\} && (\text{CCR}, a \in \mathrm{Cyl} \cup \mathfrak{X}) \\ \psi \cdot \psi' &= \psi\,\psi' && (\mathfrak{B}\text{-module}) \end{aligned} \quad + \text{ linearity}$$

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$$\begin{aligned} \omega_0(a \cdot X) &= 0 && (a \in \mathfrak{H}_{\mathrm{C}}, X \in \mathfrak{X}) \\ \omega_0(\psi_0 + \psi_{\mathrm{AP}}) &= \int_{\mathbb{R}_{\mathrm{Bohr}}} \psi_{\mathrm{AP}}\, \mathrm{d}\mu_{\mathrm{Bohr}} \end{aligned}$$

Theorem: Thiemann, Hanusch, Engle 2016; Fleischhack 2018

Assume
- $\mathfrak{A} := C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$
- $\mathfrak{B} := \{\psi \in \mathfrak{A} \mid \psi^{(n)} \in \mathfrak{A} \quad \forall n\}$

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Remark Holds also for $\mathfrak{A} := C_{\mathrm{AP}}(\mathbb{R})$

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