Conformally isometric embeddings and Hawking temperature

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19th century. Surfaces
Manifolds throughout the centuries

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- 20th century. Atlases
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- The Whitney embedding theorem: any $n$–dimensional manifold can be embedded in $\mathbb{R}^N$ as a surface, where $N$ is at most $2n$. 
A (pseudo) Riemannian curved metric $g$ on $M$ is induced from a flat metric $\eta$ on $\mathbb{R}^N$:

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Folk saying: any surface can be locally isometrically embedded in $\mathbb{R}^3$. 

Improved folk saying: The Cartan–Janet theorem (local, real analytic).

Thomas (1925), Berger, Bryant, Griffiths (1983): Holonomy obstructions and rigidity theorems if $N < n(n + 1)/2$.

The Nash–Clarke global embedding theorems ($C^3$ embeddings) $N \leq n(2n^2 + 37)/6 + 5n^2/2 + 3$ if $g$ is Lorentzian.

Embedding class = the smallest integer $N - n$.

The Schwarzschild metric: embedding class 2 (local - Kasner (1921), global - Fronsdal (1959)).

Fubini–Study metric on $\mathbb{C}P^2$: embedding class still not known (neither local nor global!). At least 3, at most 4.
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Conformal isometric embeddings

- An immersion $\iota : (M, g) \rightarrow \mathbb{R}^N$ such that $\iota^*(\eta) = \Omega^2 g$ for some $\Omega : M \rightarrow \mathbb{R}^+$, and $\iota(M) \subset \mathbb{R}^N$ is diffeomorphic to $M$. 

- The Jacobowitz–Moore thm (local, analytic): $N \leq n\left(n + \frac{1}{2}\right) - 1$.

- Naive counting: $N$ embedding functions $X_1, ..., X_N$ of local coordinates $x_1, ..., x_n$ such that $g = g_{ab}(x) dx_a dx_b$.

- $\eta_{\alpha\beta} \partial X_\alpha \partial x_a \partial X_\beta \partial x_b = \Omega^2 g_{ab}$, $\alpha, \beta = 1, ..., N$, $a, b = 1, ..., n$.

- PDEs for $(N + 1)$ unknown functions $(X_\alpha, \Omega)$ of $x_a$.

- This talk:
  1. Global conformal embedding of the Schwarzchild metric.
  2. Obstructions to conformal embeddings of class 1
  3. Hawking and Unruh temperatures.
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\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^a} \frac{\partial X^\beta}{\partial x^b} = \Omega^2 g_{ab}, \quad \alpha, \beta = 1, \ldots, N, \quad a, b = 1, \ldots, n.
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  1. Global conformal embedding of the Schwarzschild metric.
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  3. Hawking and Unruh temperatures.
Given an Einstein Lorentzian four-manifold \((M, g)\), seek an isometric embedding of \(\hat{g} = \Omega^2 g\) into \(\mathbb{R}^5\), with second fundamental form
\[
\hat{K}_{ab} = \hat{\sigma}_{ab} + \frac{1}{4} \hat{K} \hat{g}_{ab}
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- Conformal rescalling and spinors:

\[ \hat{C}^d_{abc} = C^d_{abc}, \quad \hat{\sigma}_{ab} = \Omega \sigma_{ab} \]

\[ C_{abcd} = \psi_{ABCD} \epsilon_{A'B'C'D'} + \psi_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}. \]
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**Theorem 1.** The necessary and sufficient conditions for the existence of a local conformal embedding of class 1, with the trace–free part of \(\hat{K}_{ab}\) given by \(\Omega \sigma_{ab}\) are

\[
\nabla_{A'} (A' \sigma_{BC})_{B'} = 0, \quad \sigma_{(AB} C' D') \sigma_{CD)C'D'} = \pm 2 \psi_{ABCD} \quad (*).
\]

Given a solution to \((*)\), there exists a 6D space of pairs \((\Omega, \hat{K})\).
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- **Theorem 2.** A local conformal embedding \(\iota\) of Theorem 1, such that \(\text{rank}(K_{ab})\) is maximal at some \(p \in M\), is rigid in a neighbourhood of \(p\) up to conformal transformations of \(\mathbb{R}^{r,s}, r + s = 5\).
LETTER TO THE EDITOR

Twistor equation in a curved spacetime

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Abstract. The twistor equation is studied in a four-real-dimensional spacetime. All the metric tensors which locally admit a solution are found. They either belong to the Fefferman class or are given by the Trautman-Kerr-Schild anzatz by using a non-twisting null conformal Killing vector field in the Minkowski spacetime. The corresponding solutions are derived.
Algebraic invariants of the Weyl tensor

\[ I = \psi_{ABCD} \psi^{ABC}{}^D, \quad J = \psi_{AB}^{\phantom{AB}CD} \psi_{CD}^{\phantom{CD}EF} \psi_{EF}^{\phantom{EF}AB}. \]

Algebraically special \( J^2 - 6I^3 = 0 \). Type 3, or type N: \( I = J = 0 \).
Local curvature obstructions

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- **Proposition 1.** Reality of \( I \) and \( J \) is necessary for existence of a class one conformal embedding.
- **Corollary:** the Kerr metric does not admit a class 1 conf. embedding.
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Riemannian, or neutral signature: self–dual, and anti–self–dual Weyl spinors \( C' \) and \( C \) are independent.
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- Proposition 2. The conditions
  
  \[ I = I', \quad J = J' \]

  are necessary for existence of a class one conformal embedding.

- Corollary: A Riemannian manifold with self–dual Weyl tensor admits a class one conformal embedding iff it is conformally flat.
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- Corollary: A Riemannian manifold with self–dual Weyl tensor admits a class one conformal embedding iff it is conformally flat.

- The conformal embedding class of \( \mathbb{C}P^2 \) is therefore at least two. It is known to be at most three. What is it?
Spherically symmetric conformal embedding

\[ g = V dt^2 - V^{-1} dr^2 - r^2 (d\theta^2 + \sin \theta^2 d\phi^2), \]
where \( V = V(r) \) has a finite number of simple zeroes \( r_0 > r_1 > r_2 \ldots \).
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Theorem 3. If the conformal embedding \( \iota : M \rightarrow \mathbb{R}^5 \) is global on at least one sphere of symmetry, then

- \( \sigma_{ab} \) is spherically symmetric, and
- \( \iota \) can be chosen to be spherically symmetric.

In the real analytic category the embedding depends on two arbitrary functions of one variable.

Proof: GHP formalism and harmonic analysis for part one.
Cauchy–Kowalewska for part two.

An example of a regular embedding

\[ \Omega g = dT^2 - dX^2 - dR^2 - R^2 (d\theta^2 + \sin \theta^2 d\phi^2). \]

Set \( \Omega = \mathbb{R} / r. \)
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  \[ \Omega^2 g = dT^2 - dX^2 - dR^2 - R^2(d\theta^2 + \sin \theta^2 d\phi^2) \]. Set \( \Omega = R/r \).
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\[ g = V dt^2 - V^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \] where \( V = V(r) \) has a finite number of simple zeroes \( r_0 > r_1 > r_2 \ldots \).

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\[ \Omega^2 g = dT^2 - dX^2 - dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2). \] Set \( \Omega = R/r \).

Find an isometric embedding of \( r^{-2}(V^{-1} dr^2 - V dt^2) \) in \( AdS_3 \)

\[ \frac{dR^2 + dX^2 - dT^2}{R^2}. \]
The unique static, spherically symmetric, global conformal embedding.
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\[ T = \sinh (ta) \frac{h(r)}{ar} \sqrt{V(r)}, \quad X = \cosh (ta) \frac{h(r)}{ar} \sqrt{V(r)}, \quad R = h(r), \quad \text{where} \]

\[ h = \exp \left( \int \frac{V(2V - rV') \pm ar \sqrt{V(4V + 4a^2r^2 - (2V - rV')^2)}}{2rV(a^2r^2 + V)} dr \right). \]
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Regularity at a zero \( r = \bar{r} \) of \( V \): \( a = \pm \frac{1}{2} V' |_{r=\bar{r}} \) (the surface gravity).
Global conformal embedding of Schwarzschild

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h = \exp \left( \int \frac{V(2V - rV') \pm ar \sqrt{V(4V + 4a^2r^2 - (2V - rV')^2)}}{2rV(a^2r^2 + V)} \, dr \right).
\]

- Regularity at a zero \( r = \bar{r} \) of \( V \): \( a = \pm \frac{1}{2} V'\big|_{r=\bar{r}} \) (the surface gravity).
- If \( V \to 1 \) as \( r \to \infty \), then \( R \sim r \) and \( \Omega \sim 1 \) as \( r \to \infty \).
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\[ h = \exp \left( \int \frac{V(2V - rV') \pm ar \sqrt{V(4V + 4a^2r^2 - (2V - rV')^2)}}{2rV(a^2r^2 + V)} dr \right). \]

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If \( V \to 1 \) as \( r \to \infty \), then \( R \sim r \) and \( \Omega \sim 1 \) as \( r \to \infty \).

\( V = 1 - 2m/r, 
R(r) = \exp (\int \frac{p}{q} dr), \)

where

\[ p = 48m^3 - 16m^2r - r^{3/2} \sqrt{r^3 + 2mr^2 + 4m^2r + 72m^3}, \]
\[ q = (32m^3 - 16m^2r - r^3)r. \]
Theorem 4. Let \((I_\pm)^{Schw}\) and \((I_\pm)^5\) be null infinities of the compactified Schwarzschild \(\overline{M}\), and the compactified Minkowski \(\overline{\mathbb{R}^{4,1}}\). The conformal embedding extends to a map \(\iota: \overline{M} \rightarrow \overline{\mathbb{R}^{4,1}}\) s. t. \(\iota((I_\pm)^{Schw}) = p_\pm\) where \(p_- \in (I_-)^5\) and \(p_+ \in (I_+)^5\) are points with coordinates \((0, N \subset S^3)\).
The Hawking effect: a temperature measured by asymptotic observers is $T_H = \kappa / 2\pi$, where

$$\nabla_a (|K|^2) = -2\kappa K_a,$$

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HAWKING TO UNRUH

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- The surface gravity is conformally invariant as long as $\Omega$ and $d\Omega$ are regular on the horizon, $\Omega$ is static, and $\Omega \to 1$ when $r \to \infty$.

- A trajectory of $K = \partial/\partial t$ in $M$ lifts to a hyperbola in $\mathbb{R}^4$,
  \[
  X_1^2 - X_0^2 = 16 \mu^2 (r^2) - \frac{r^2}{1 - 2\mu r} \equiv \alpha - 2\mu r,
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  where $r$ fixed.

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Open problems

- Extend to higher-dimensional black-holes, and use to study the causal properties of asymptotically flat space-times (Peter Cameron, in progress).

- Develop the rigidity theory of conformal embeddings of classes between $2$ and $\frac{n(n+1)}{2}−1$ (Cartan–Kähler theory, prolongations).

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- Find (or rule out!) a conformal isometric embedding of $\mathbb{CP}^2$ in $\mathbb{R}^6$.

- Embeddings (isometric, conformal) of gravitational instantons: Eguchi–Hanson, self-dual Taub NUT can be explicitly isometrically embedded in $\mathbb{R}^8$, and can not be isometrically embedded in $\mathbb{R}^6$. What is their embedding class?

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