Conformally isometric embeddings and Hawking temperature

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and
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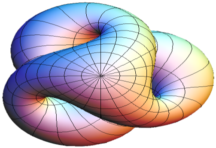
University of Cambridge.

- Maciej Dunajski, Paul Tod (2019) *Conformally isometric embeddings and Hawking temperature*, `arXiv: 1812.05468`, CQG 2019.
- Maciej Dunajski, Paul Tod (2019) *Conformal and isometric embeddings of Gravitational Instantons*, Preprint.

- 19th century. Surfaces



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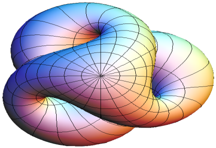

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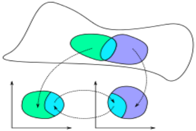

Manifolds throughout the centuries

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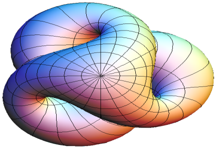

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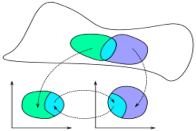

- The Whitney embedding theorem: any n–dimensional manifold can be embedded in $\mathbb{R}^N$ as a surface, where N is at most $2n$.

Isometric embeddings

- A (pseudo) Riemannian curved metric g on M is induced from a flat metric η on $\mathbb{R}^N$:

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  2. Fubini–Study metric on $\mathbb{CP}^2$: embedding class still not known (neither local not global!). At least 3, at most 4.

- An immersion $\iota : (M, g) \to \mathbb{R}^N$ such that $\iota^*(\eta) = \Omega^2 g$ for some $\Omega : M \to \mathbb{R}^+$, and $\iota(M) \subset \mathbb{R}^N$ is diffeomorphic to M.

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$$\eta_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^a} \frac{\partial X^\beta}{\partial x^b} = \Omega^2 g_{ab}, \quad \alpha, \beta = 1, \dots, N, \quad a, b = 1, \dots, n.$$

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  3. Hawking and Unruh temperatures.

Class 1 conformal embeddings

- Given an Einstein Lorentzian four-manifold (M, g), seek an isometric embedding of $\hat{g} = \Omega^2 g$ into $\mathbb{R}^5$, with second fundamental form

$$\hat{K}_{ab} = \hat{\sigma}_{ab} + \frac{1}{4}\hat{K}\hat{g}_{ab}$$

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$$\nabla_{A'(A}\sigma_{BC)B'}{}^{A'} = 0, \;\; \sigma_{(AB}{}^{C'D'}\sigma_{CD)C'D'} = \pm 2\psi_{ABCD} \quad (*).$$

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- Theorem 2. A local conformal embedding ι of Theorem 1, such that $\mathrm{rank}(K_{ab})$ is maximal at some $p \in M$, is rigid in a neighbourhood of p up to conformal transformations of $\mathbb{R}^{r,s}$, $r + s = 5$.

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LETTER TO THE EDITOR

Twistor equation in a curved spacetime

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Abstract. The twistor equation is studied in a four-real-dimensional spacetime. All the metric tensors which locally admit a solution are found. They either belong to the Fefferman class or are given by the Trautman–Kerr–Schild anzatz by using a non-twisting null conformal Killing vector field in the Minkowski spacetime. The corresponding solutions are derived.

Local curvature obstructions

- Algebraic invariants of the Weyl tensor

$$I = \psi_{ABCD}\psi^{ABCD}, \quad J = \psi_{AB}{}^{CD}\psi_{CD}{}^{EF}\psi_{EF}{}^{AB}.$$

Algebraically special $J^2 - 6I^3 = 0$. Type 3, or type N: $I = J = 0$.

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are necessary for existence of a class one conformal embedding.
- Corollary: A Riemannian manifold with self–dual Weyl tensor admits a class one conformal embedding iff it is conformally flat.
- The conformal embedding class of $\mathbb{CP}^2$ is therefore at least two. It is known to be at most three. What is it?

Spherically symmetric conformal embedding

- $g = V dt^2 - V^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2)$, where $V = V(r)$ has a finite number of simple zeroes $r_0 > r_1 > r_2 \dots$.

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- An example of a regular embedding

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- Find an isometric embedding of $r^{-2}(V^{-1}dr^2 - Vdt^2)$ in AdS_3

$$\frac{dR^2 + dX^2 - dT^2}{R^2}.$$

- The unique static, spherically symmetric, global conformal embedding.

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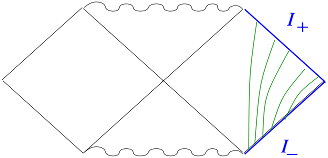
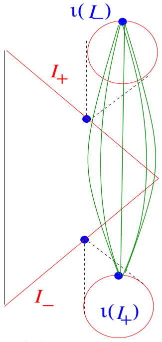
- $V = 1 - 2m/r$, $R(r) = \exp\left(\int \frac{p}{q} dr\right)$, where

$$\begin{aligned} p &= 48m^3 - 16m^2r - r^{3/2}\sqrt{r^3 + 2mr^2 + 4m^2r + 72m^3}, \\ q &= (32m^3 - 16m^2r - r^3)r. \end{aligned}$$

- Theorem 4. Let $(\mathcal{I}_\pm)^{Schw}$ and $(\mathcal{I}_\pm)^5$ be null infinities of the compactified Schwarzschild $\overline{M}$, and the compactified Minkowski $\overline{\mathbb{R}}^{4,1}$. The conformal embedding extends to a map $\iota : \overline{M} \to \overline{\mathbb{R}}^{4,1}$ s. t. $\iota((\mathcal{I}_\pm)^{Schw}) = p_\pm$ where $p_- \in (\mathcal{I}_-)^5$ and $p_+ \in (\mathcal{I}_+)^5$ are points with coordinates $(0, N \subset S^3)$.



(M, g)



$(R^{4,1}, \eta)$

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Happy Birthday Jurek!