Isolated horizons, near horizon geometries and the Petrov type D equation

*1, 2). **Denis Dobkowski-Ryłko**, J. Lewandowski, T. Pawłowski (2018);*
3). JL, A. Szereszewski (2018);
4). DDR, W. Kamiński, JL, AS (2018);
5). DDR, JL, I. Rácz (2019);
6). M. Kolanowski, JL, AS (2019).

Jurekfest, Warsaw, September 16-20, 2019

Plan of the talk

1). Non-extremal Isolated Horizons stationary to the second order:

- type D equation
- solution to the type D equation on axisymmetric 2-sphere section of the IH
- solution to the type D equation on genus>0 section of the IH
- solution to the type D equation on IHs of the non-trivial topology;
- bifurcated Petrov type D horizon

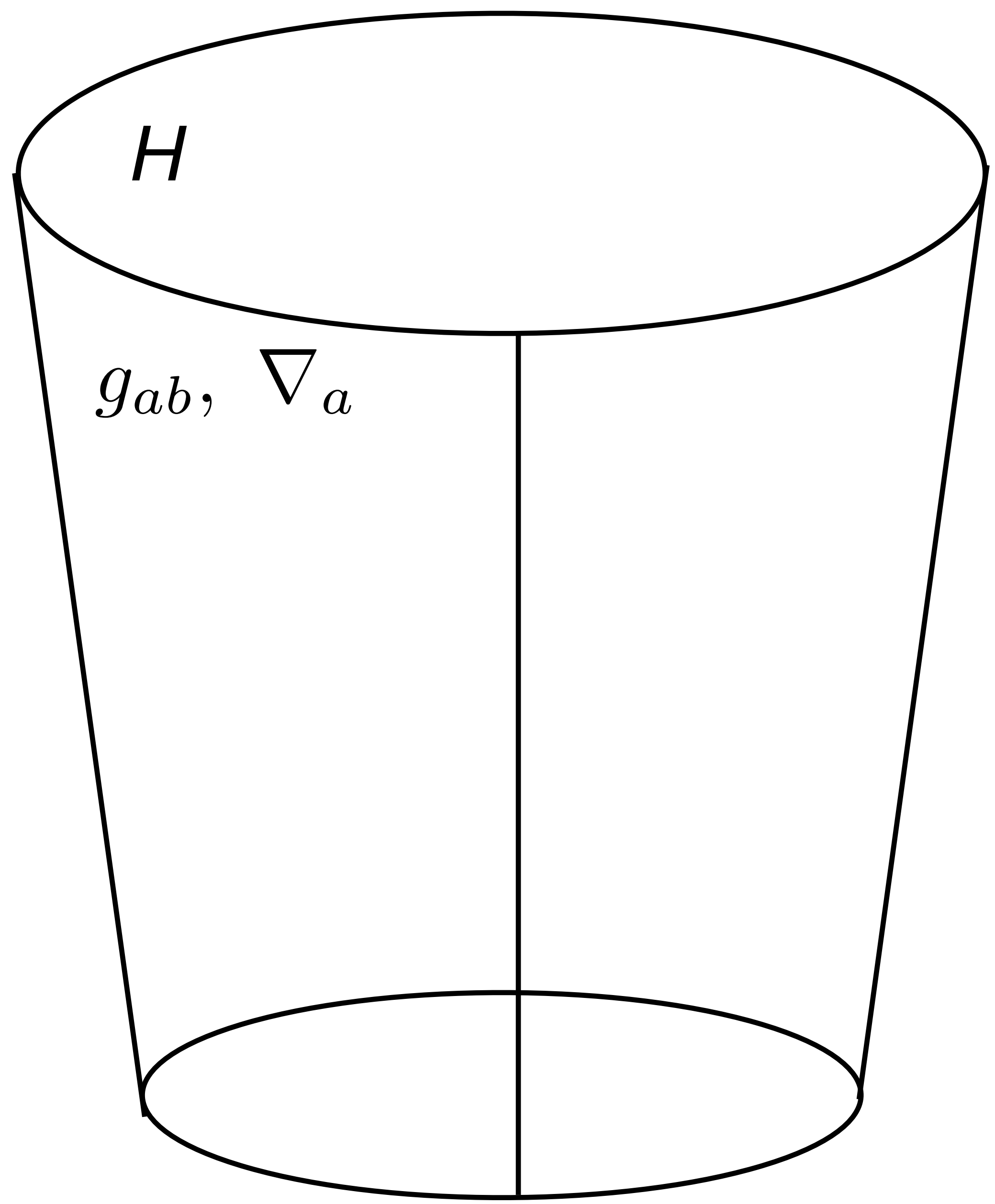
2). Extremal Isolated Horizons stationary to the second order and the NHG:

- Near Horizon Geometry equations in (n+2)-dim
- NHG in 4-dim:
    - Solution to NHG for genus=0
    - Solution to NHG for genus>0
- Spacetimes foliated by non-expanding surfaces of co-dimension 1

Non-extremal Isolated Horizons stationary to the second order

Isolated Horizons (stationary to the 2nd order)

H - 3dim null surface in 4dim spacetime M



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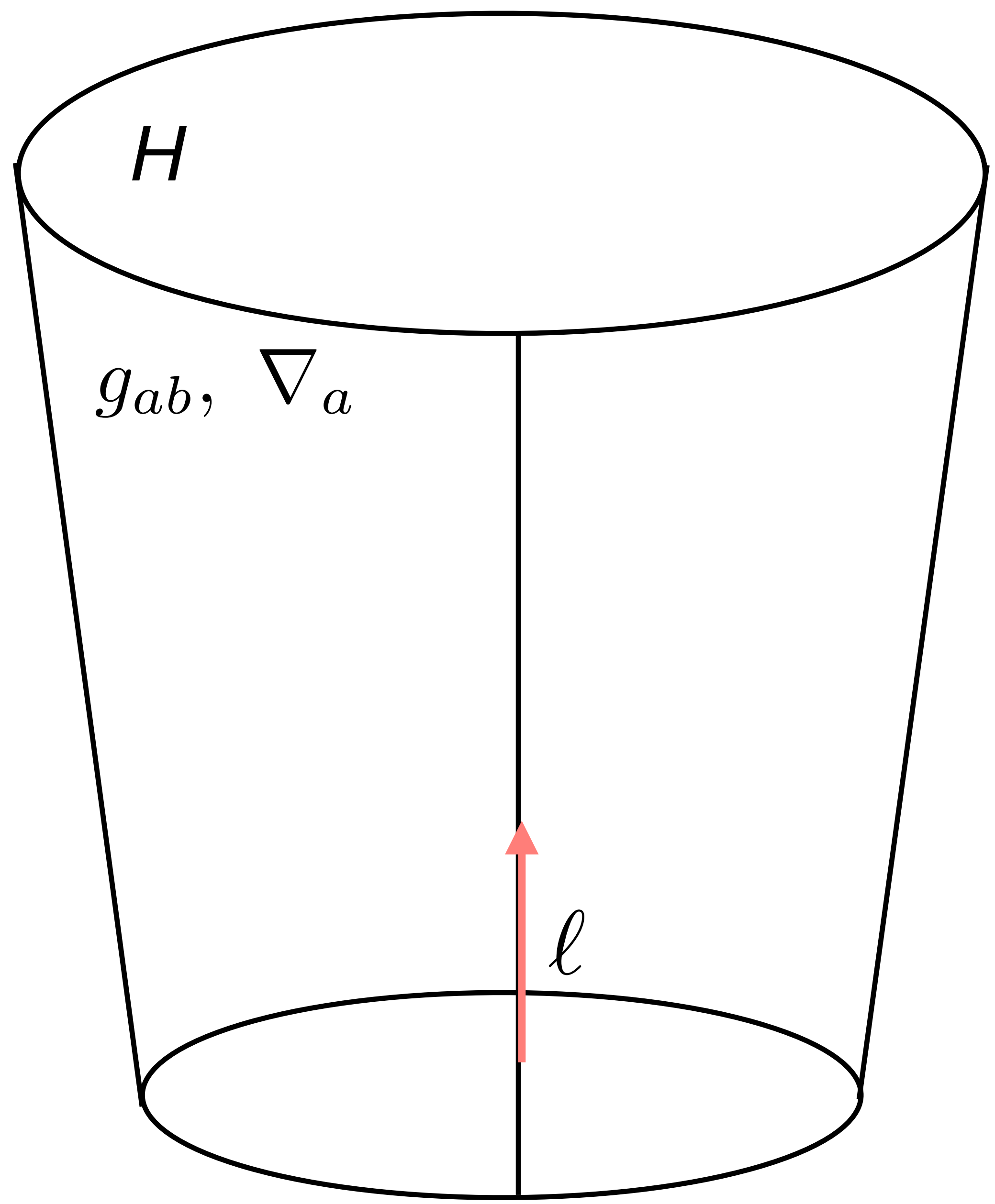
$$\ell^{\mu}\ell_{\mu} = 0$$



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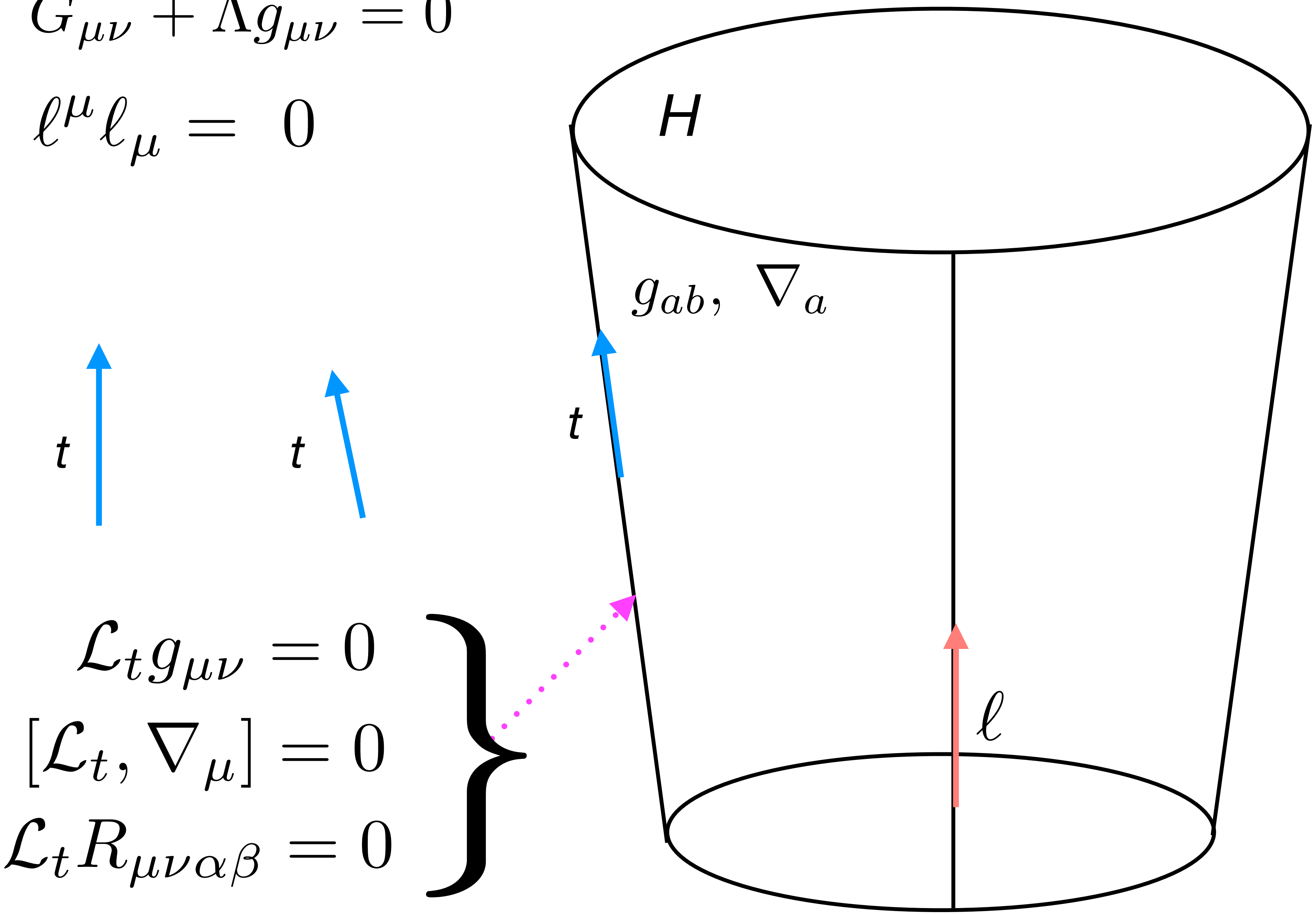

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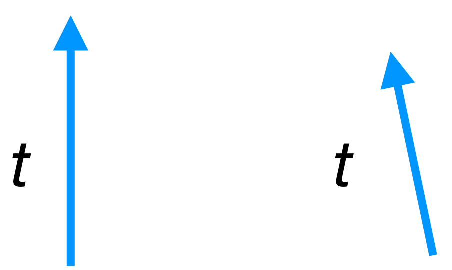

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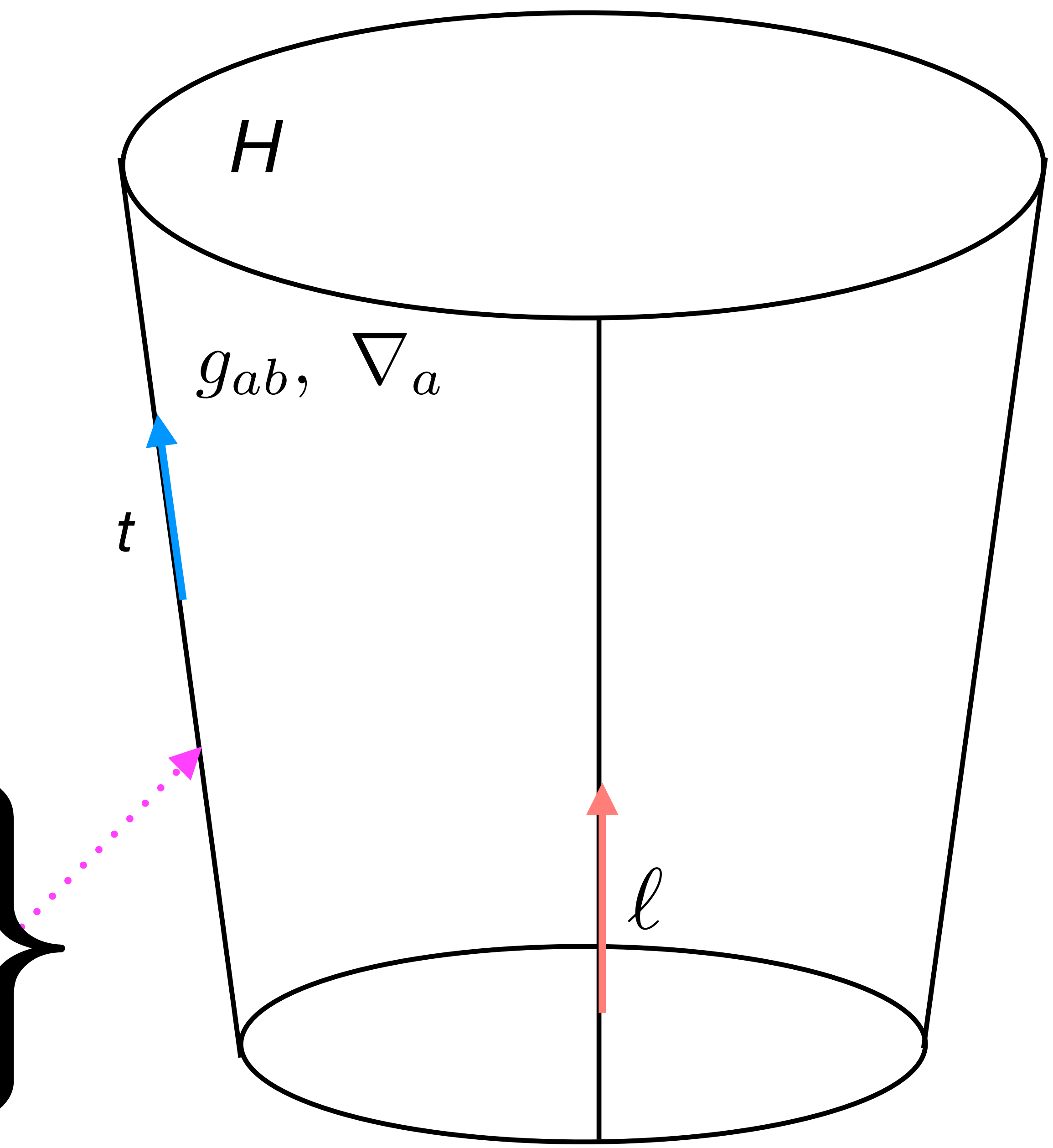

Rotation Potential:

$$\nabla_a \ell^b = \omega_a \ell^b$$

Surface Gravity:

$$\kappa^{\ell} = \omega_a \ell^a$$

Non-extremality condition:

$$\kappa^{\ell} \neq 0$$

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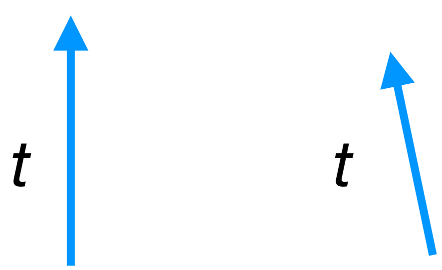

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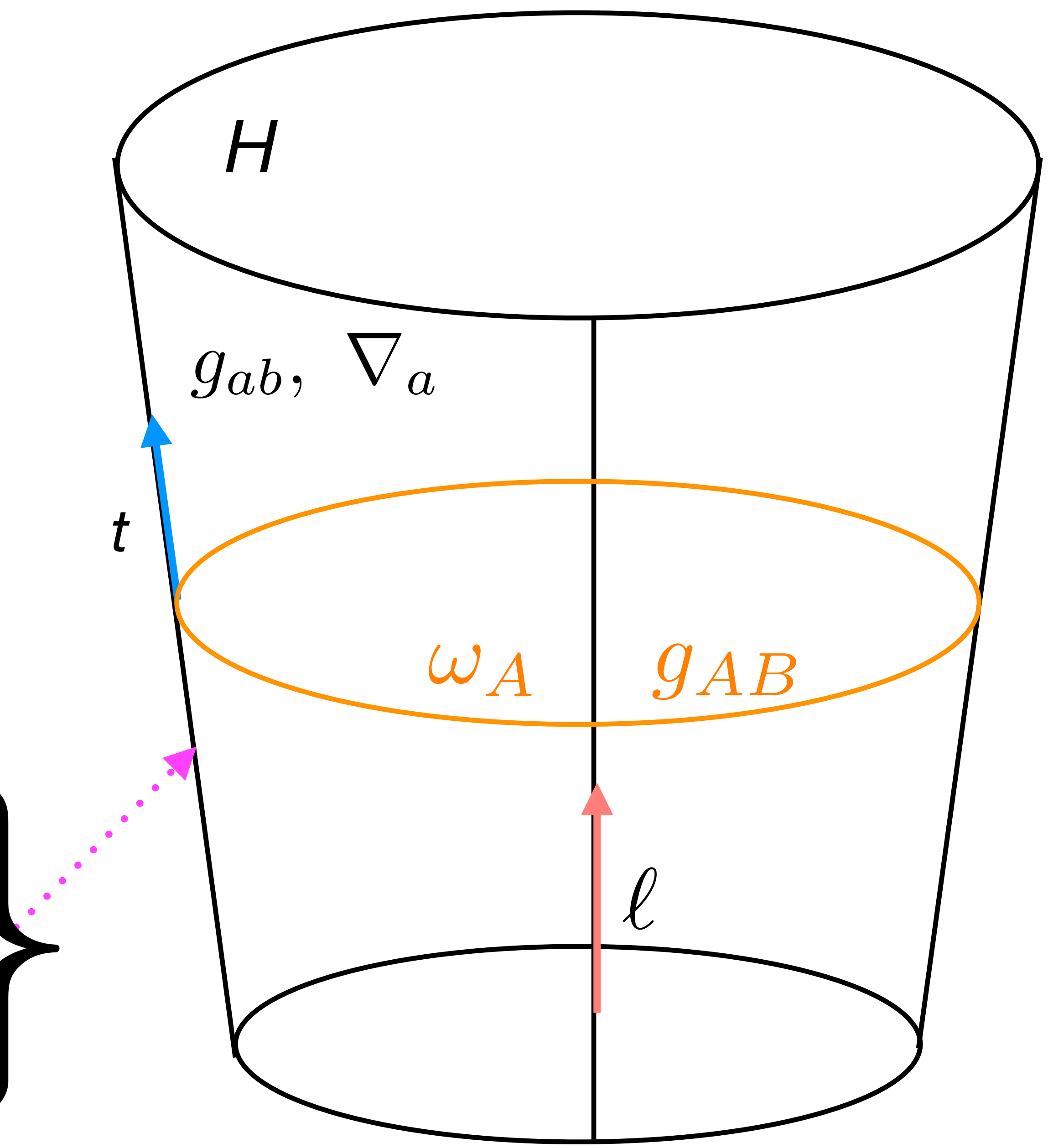

$$(\omega_A, g_{AB})$$

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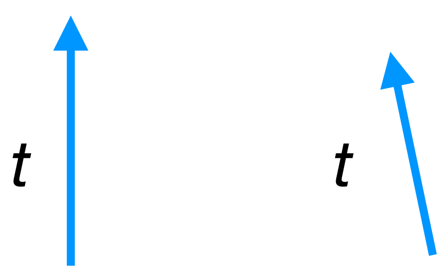

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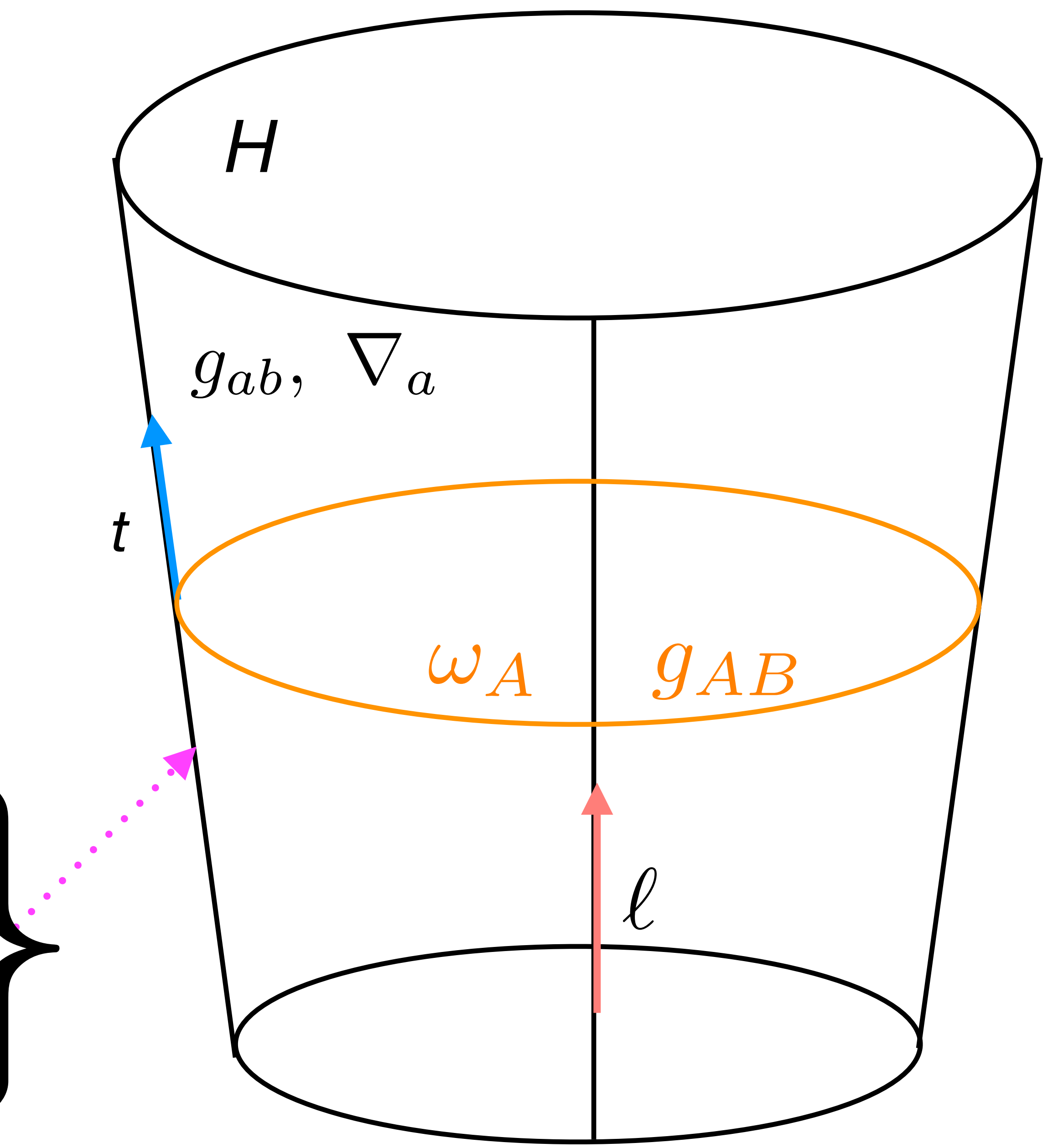

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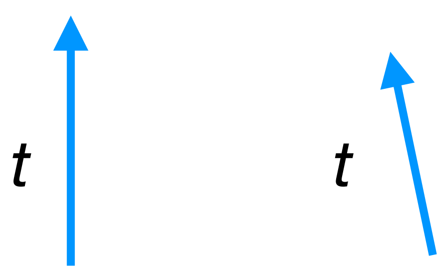

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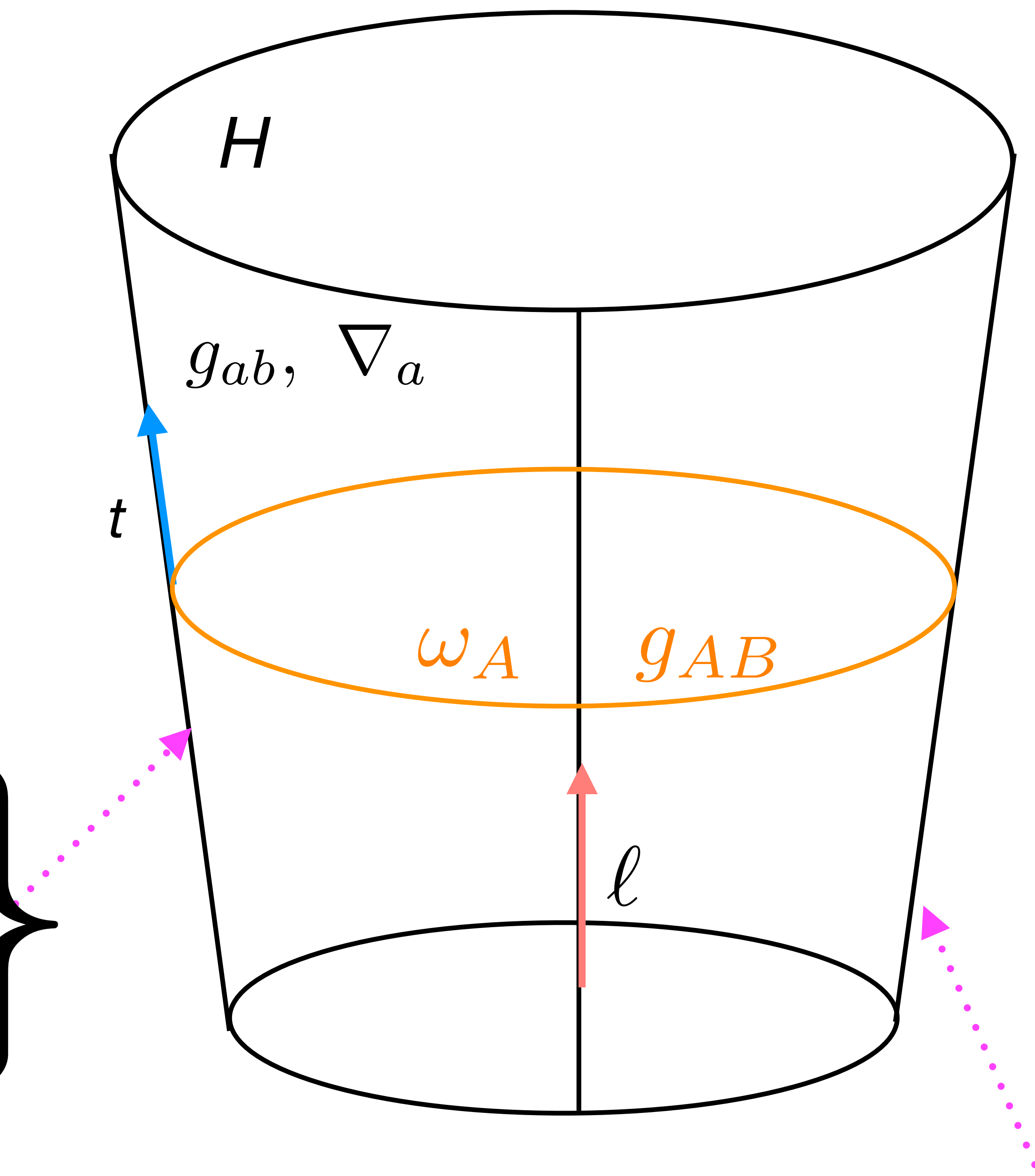

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$$(\omega_A, g_{AB}) \rightarrow (\omega_a, g_{ab}) \rightarrow g_{\mu\nu} \ , \ \nabla_\mu \ , \ R_{\mu\nu\alpha\beta}$$

The type D equation

The Weyl tensor in Newman-Penrose formalism

- Spacetime Weyl tensor in the null frame formalism may be expressed by the following complex valued N-P components:

$$\Psi_0 = C_{4141}, \quad \Psi_1 = C_{4341} \quad \Psi_2 = C_{4123},$$
$$\Psi_3 = C_{3432}, \quad \Psi_4 = C_{3232}$$

- Four components are constant along the null generators of H:

$$D\Psi_I = 0, \qquad I = 0, 1, 2, 3$$

- Stationarity to the second order: $D\Psi_4 = 0$

- The components Ψ_0 and Ψ_1 vanish due to vanishing of the expansion and shear of ℓ :

$$\Psi_0 = \Psi_1 = 0$$

- $\Psi_2 = -\frac{1}{2}(K + i\Omega) + \frac{1}{6}\Lambda =: \Psi + \frac{1}{6}\Lambda$ where $\Omega\eta_{AB} = d\omega_{AB}$

Possible Petrov types

The spacetime Weyl tensor at H is determined by the data

$$(S, g_{AB}, \omega_A)$$

Theorem 1

The possible Petrov types of *H* are:

I, II, D, III, N, O

DDR, Lewandowski, Pawłowski (2018)

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$\Psi_2 \neq 0 \quad \Rightarrow$ **generically type II, unless…**

DDR, Lewandowski, Pawłowski (2018)

The Petrov type D equation

We use a null 2-frame

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad \eta_{AB} = i(\bar{m}_A m_B - \bar{m}_B m_A)$$

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At *H* the spacetime Weyl tensor is of the Petrov type D iff the following two conditions are satisfied:

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DDR, Lewandowski, Pawłowski (2018)

Solution to the type D equation on axisymmetric 2-sphere section of IH

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Consider a metric: $g_{AB}dx^A dx^B = Q^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2)$

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Type D equation reads: $\partial_x^2 \Psi_2^{-\frac{1}{3}} = 0$

and its solution: $(c_1 x + c_2)^{-3} = \Psi_2 = -\frac{1}{2}(K + i\Omega) + \frac{1}{6}\Lambda$

Axisymmetric solution to the Petrov type D equation

Theorem 3

The family of axisymmetric solutions to the Petrov type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by two numbers *(A,J)*: the area and angular momentum, respectively. They can take the following values:

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Lewandowski, Pawłowski (2003) for $\Lambda = 0$

Embeddability of the axisymmetric solutions

Every solution defines a type D isolated horizon whose intrinsic geometry coincides with the intrinsic geometry of a non-extremal Killing horizon contained in one of the following spacetimes:

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1). **Kerr - (anti) de Sitter**;

2). **Schwarzschild - (anti) de Sitter**;

3). **Near horizon limit spacetime** near an extremal horizon contained either in the K(a)dS or S(a)dS spacetime.

Solution to the type D equation on genus>0 sections of IH

Petrov type D equation on *S* of genus > 0

Consider the following metric tensor:

$$g_{AB}dx^A dx^B = \frac{2}{P^2}dz d\bar{z} \qquad m^A\partial_A = P\partial_z$$

$$\eta = i\frac{1}{P^2}dz \wedge d\bar{z}$$

Petrov type D equation reads:

$$\partial_{\bar{z}}\left(P^2\partial_{\bar{z}}\Psi_2^{-\frac{1}{3}}\right) = 0 \Rightarrow P^2\partial_{\bar{z}}\Psi_2^{-\frac{1}{3}} = F(z)$$

Globally defined holomorphic vector field: $F(z)\partial_z$

$$F(z) = F_0 = const \qquad \text{for genus} = 1;$$

$$F(z) = 0 \qquad \text{for genus} > 1.$$

Solution to the Petrov type D equation on *S* of genus > 0

It is straightforward to show that even for genus = 1:

$$F_0 \int_S \eta = i \int_S \partial_{\bar{z}} \Psi_2^{-\frac{1}{3}} dz \wedge d\bar{z} = -i \int_S d\left(\Psi_2^{-\frac{1}{3}} dz\right) = 0$$

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Theorem 4

Suppose S is a compact 2-surface genus > 0. The only solutions to the Petrov type D equation with a cosmological constant Λ **are** (g, ω) **such that:**

$$d\omega = 0 \qquad\qquad K = const \neq \frac{\Lambda}{3}$$

DDR, Kamiński, Lewandowski, Szereszewski (2018)

Solution to the type D equation on IHs of the non-trivial topology

The type D equation for $S = S_2$ and non-trivial bundle

Now: $$\int_S \Omega d\mathrm{Area} = 2\pi\kappa m =: 2\pi n \neq 0$$

where: $m \in \mathbb{Z}$

m characterizes *U(1)* bundle: $$\begin{array}{c} H \\ \downarrow \Pi \\ S \end{array}$$

DDR, Lewandowski, Rácz (2019)

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We found all axisymmetric solutions which for every value of the topological charge m set a 3-dim family that can be parametrized by the area, surface gravity and a parameter corresponding to rotation.

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Embeddabilty???

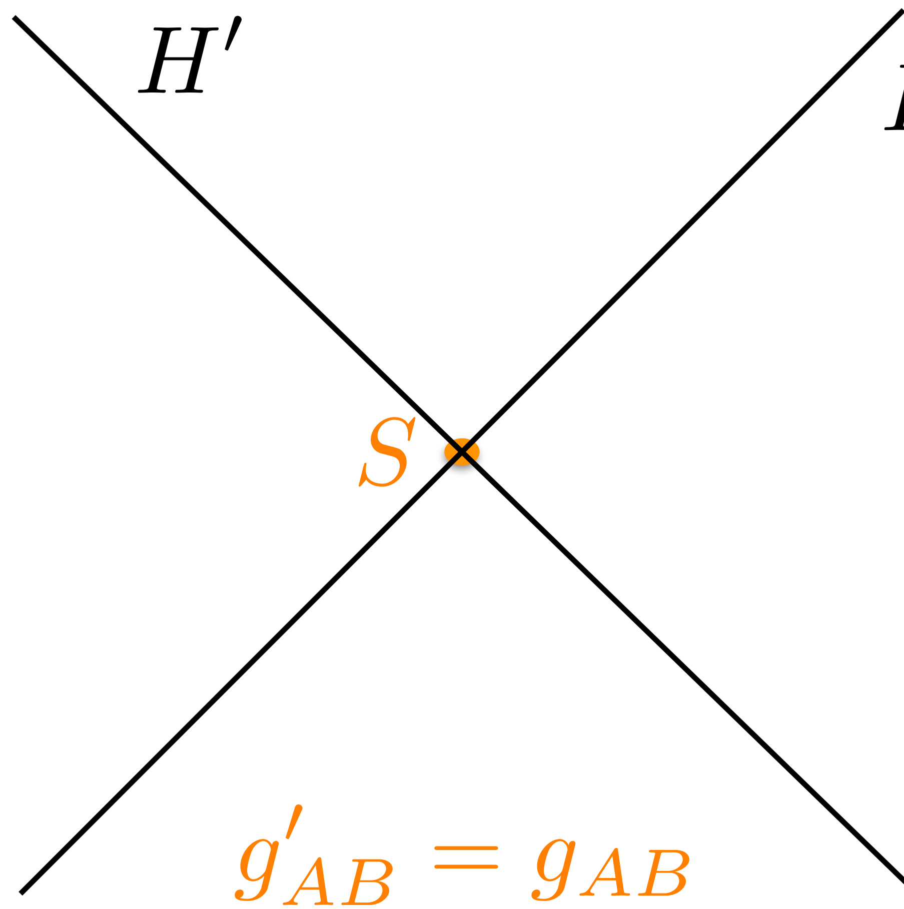
DDR, Lewandowski, Rácz (2019)

Axisymmetric solutions to the type D equation on horizons of non-trivial bundle topology

Class 1	Class 2	Class 3
$R^2 > 0$	$R^2 = \frac{1}{2\Lambda'}$	$R^2 = \frac{1}{2}\frac{\gamma}{\Lambda'\gamma - 1} \neq \frac{1}{2\Lambda'}$
$P^2 = 1 - x^2$	$P^2 = 1 - x^2$	$P^2 = \frac{(1-x^2)\left(\left(x - \frac{1}{2}\eta n\,(1-\Lambda'\gamma)\right)^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma}\right)}{\left(x - \frac{1}{2}\eta n\,(1-\Lambda'\gamma)\right)^2 + \eta^2}$
$\Omega = \frac{n}{2R^2}$	$\Omega = -\frac{2\alpha\left(1-\left(\frac{n\Lambda'}{2\alpha}\right)^2\right)^2}{\left(x - \frac{n\Lambda'}{2\alpha}\right)^3}$	$\Omega = Im\left[\frac{2i\left(1-\eta^2\left(\frac{1}{2}n\,(\Lambda'\gamma-1)+i\right)^2\right)}{\eta\gamma\left(x+\frac{1}{2}\eta n\,(\Lambda'\gamma-1)+i\eta\right)^3}\right]$

Bifurcated Petrov type D horizon

Bifurcated Petrov type D horizon: data



$$g'_{AB} = g_{AB}$$

$$\omega'_A = -\omega_A$$

$$\Psi' = \bar{\Psi}$$

Rácz
Lewandowski
Szereszewski

Bifurcated Petrov type D horizon: equations

Petrov type D equations:

- for H: $$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

- for H’: $$m^A m^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

hold simultaneously on S $\Rightarrow$ additional (axial) symmetry

Cole, Rácz, Valiente Kroon 2018
Lewandowski, Szereszewski 2018

Bifurcated Petrov type D horizon in conformally flat coordinates

$$g_{AB}dx^A dx^B = \frac{2}{P^2} dz d\bar{z} \qquad m^A\partial_A = P\partial_z$$

$$\partial_{\bar{z}}(P^2\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{F(z)}{P^2}$$

$$\partial_z(P^2\partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}}) = 0 \quad \Rightarrow \quad \partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \frac{\bar{G}(\bar{z})}{P^2}$$

$$\partial_z\partial_{\bar{z}}(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} = \partial_{\bar{z}}\partial_z(\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \quad \Rightarrow \quad \partial_z\left(\frac{F(z)}{P^2}\right) = \partial_{\bar{z}}\left(\frac{\bar{G}(\bar{z})}{P^2}\right)$$

$$\Phi := F(z)\partial_z - \bar{G}(\bar{z})\partial_{\bar{z}}$$

$$\Rightarrow \quad \mathcal{L}_\Phi g_{AB} = 0$$

$$\Rightarrow \quad \mathcal{L}_\Phi \, d\omega = 0$$

Axial symmetry without the rigidity theorem

Theorem 6

Suppose (g_{AB}, ω_A) defined on S satisfy the Petrov type D equation:

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

and the conjugate one:

$$m^A m^B \nabla_A \nabla_B (\Psi + \frac{1}{6}\Lambda)^{-\frac{1}{3}} = 0$$

Then there is a vector field Φ on S such that

$$\mathcal{L}_\Phi g_{AB} = 0 \qquad\qquad \mathcal{L}_\Phi d\omega = 0$$

$$\Phi^A = Re/Im \left(dArea^{AB} \partial_A (\Psi + \frac{\Lambda}{6})^{-\frac{1}{3}} \right)$$

Corollary: the axial symmetry for $S = S_2$

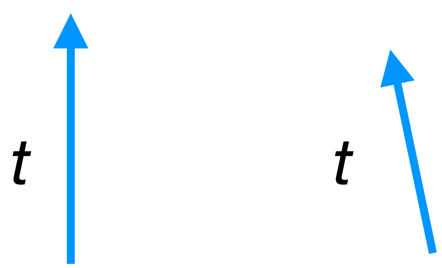
Extremal Isolated horizons stationary to the second order

Extremal Isolated Horizons to the 2nd order in (n+2)-dim spacetime

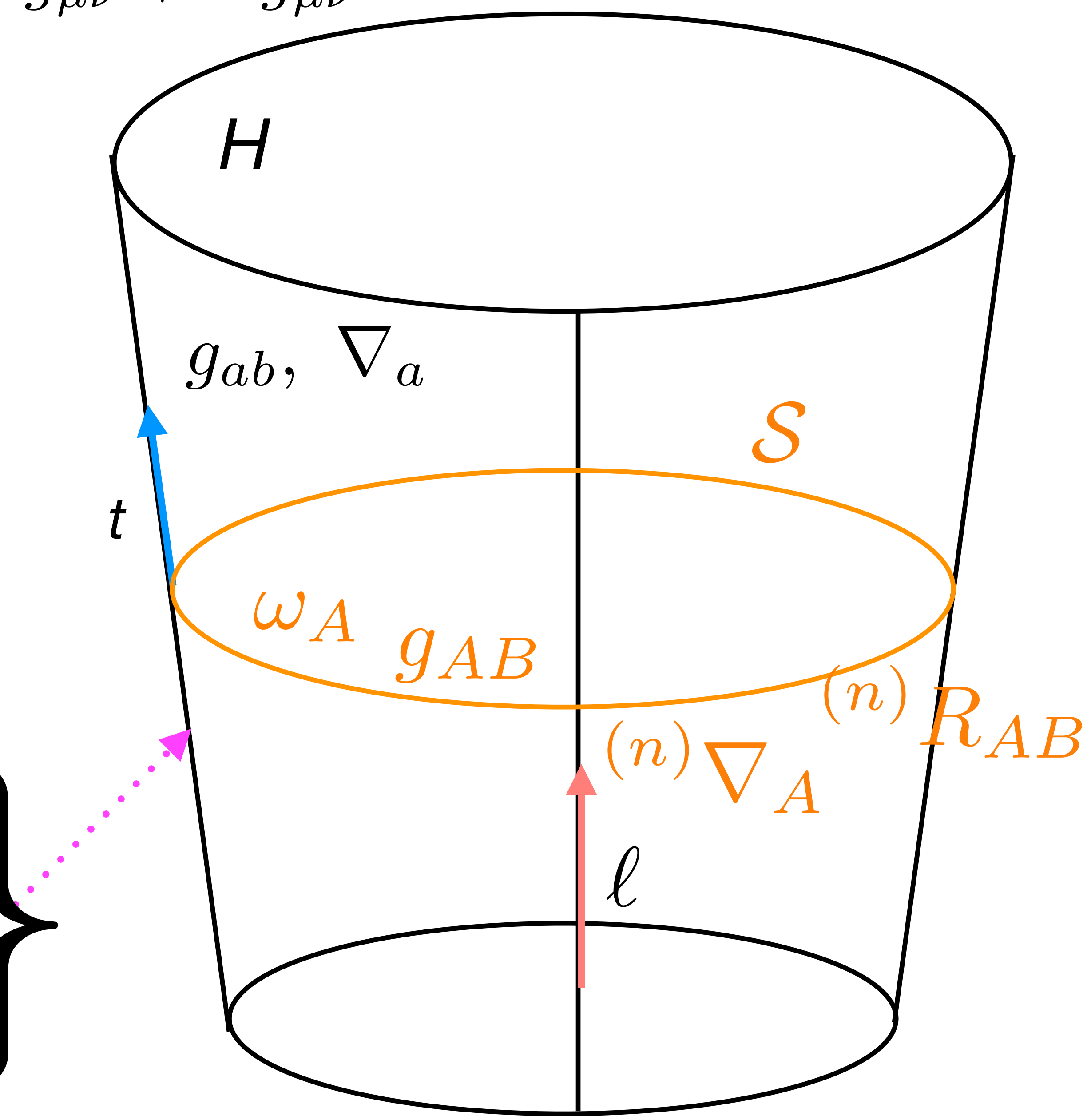
H - (n+1)-dim null surface in (n+2)-dim spacetime M

$$ {}^{(n+2)}R_{\mu\nu} - \frac{1}{2}\,{}^{(n+2)}R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0 $$

$$ \ell^{\mu}\ell_{\mu} = 0 $$



$$ \left.\begin{array}{r} \mathcal{L}_t g_{\mu\nu} = 0 \\ [\mathcal{L}_t, \nabla_\mu] = 0 \\ \mathcal{L}_t R_{\mu\nu\alpha\beta} = 0 \end{array}\right\} $$



Rotation Potential:

$$ \nabla_a \ell^b = \omega_a \ell^b $$

Surface Gravity:

$$ \kappa^{\ell} = \omega_a \ell^a $$

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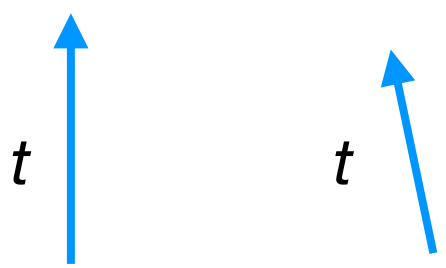

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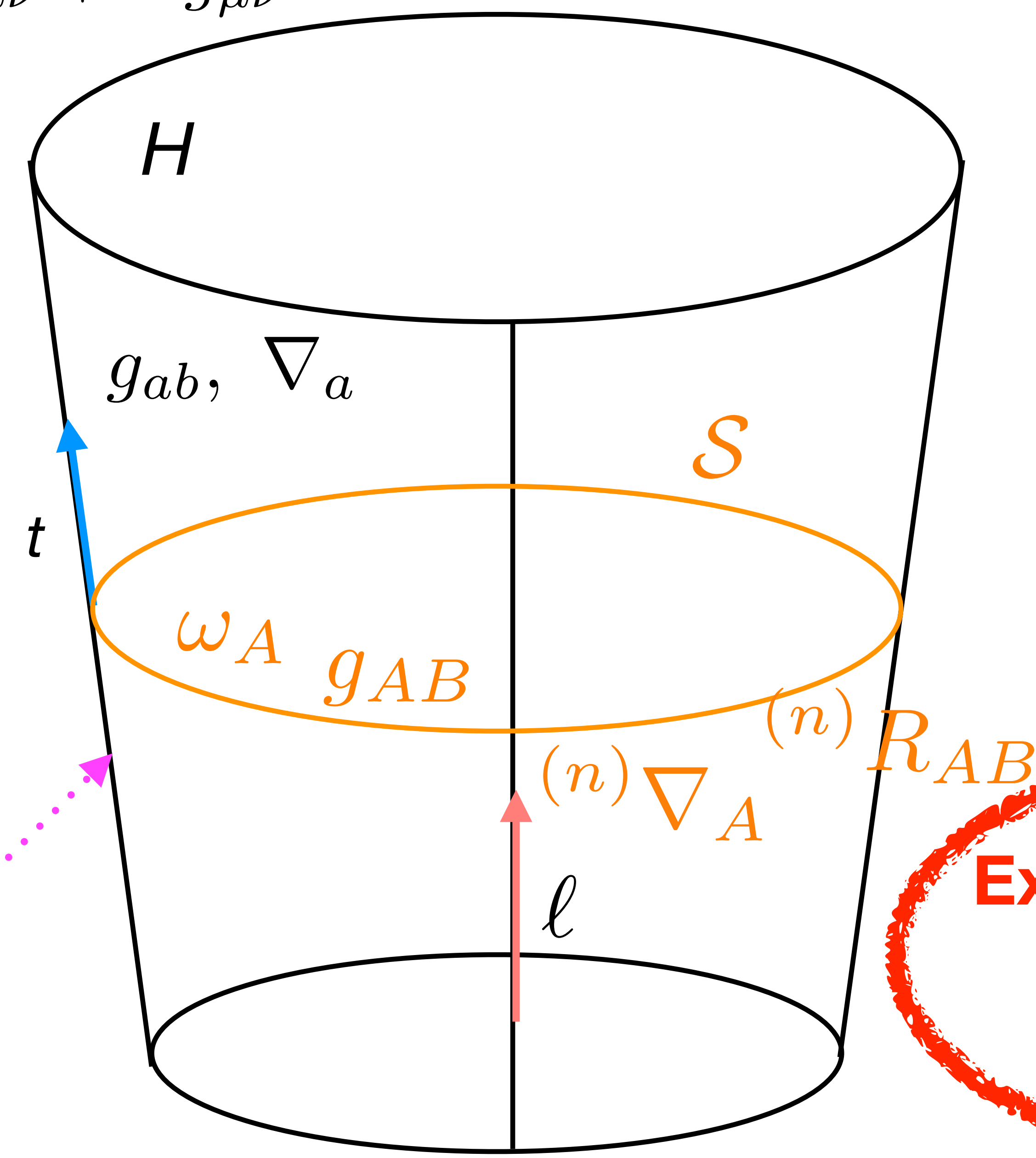

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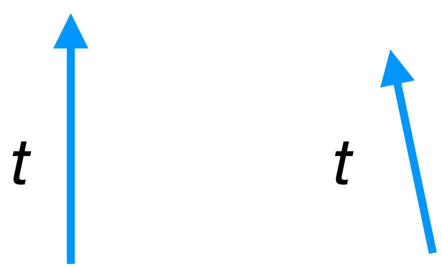

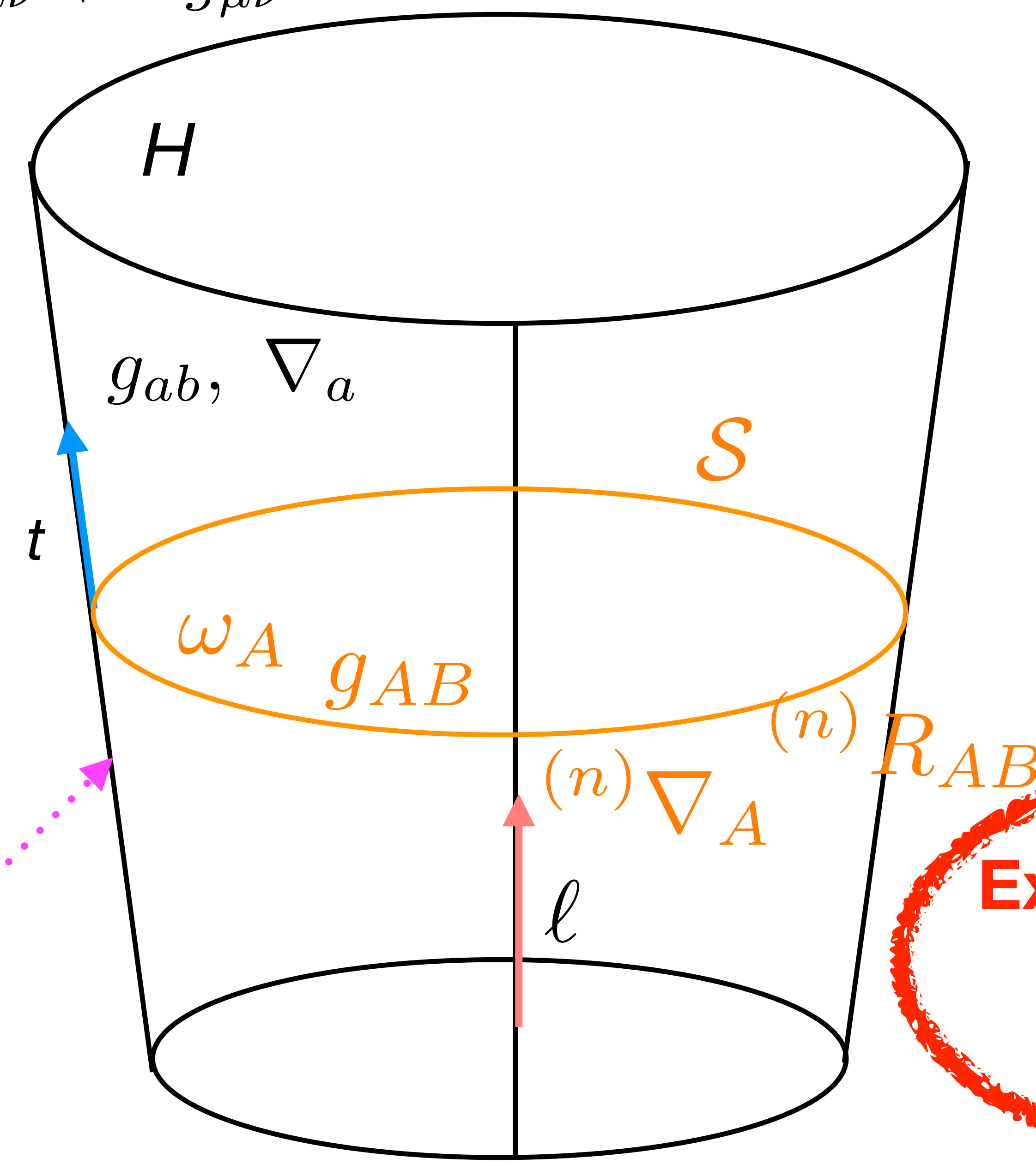

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$$\kappa^{\ell} = 0$$

$v : H \to \mathbb{R}$ **:** $\quad v|_S = const$, $\quad \ell(v) = 1$

Extremal Isolated Horizons to the 2nd order in (n+2)-dim spacetime

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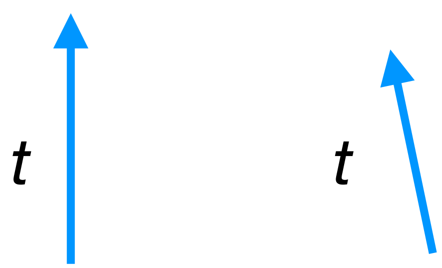

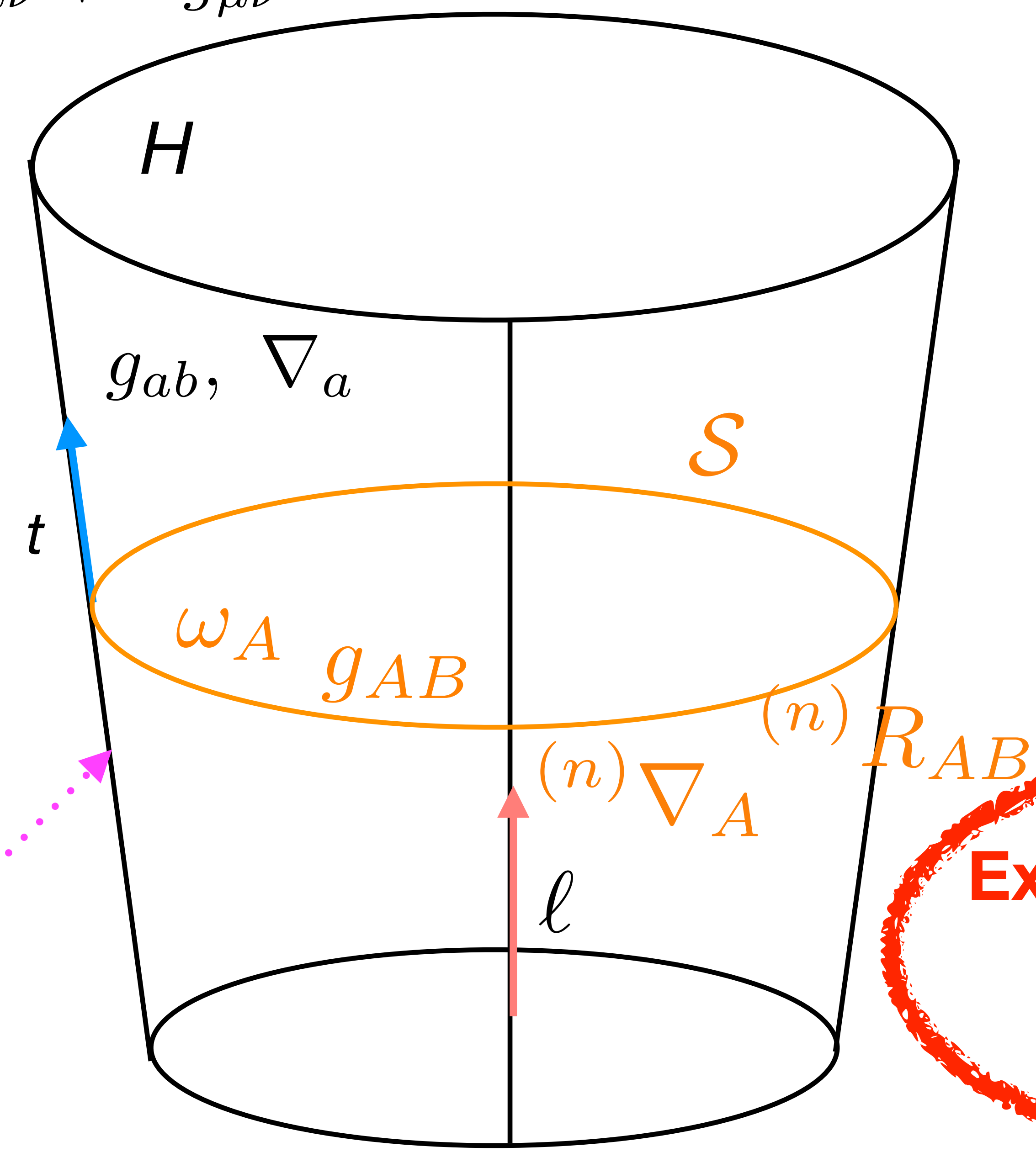

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Extremal Isolated Horizons to the 2nd order in (n+2)-dim spacetime

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t t

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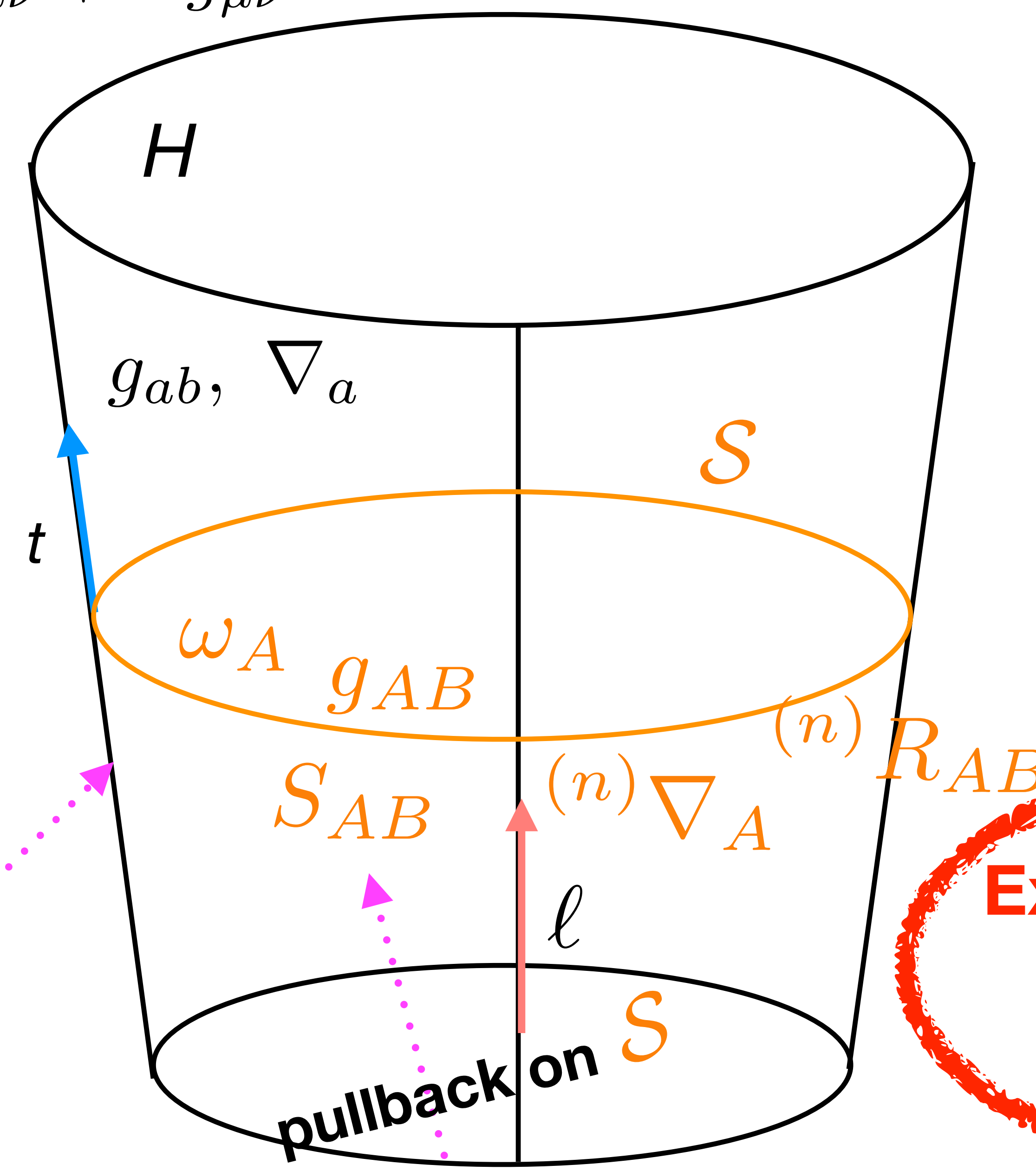

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Near Horizon Geometry equations in (n+2)-dim

Extremal Isolated Horizons to the 2nd order: equations

$$ {}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0 $$

Hajicek 1970's, Isenberg, Moncrief 1983, Ashtekar, Beetle, Lewandowski 2001

$$ S_{AB;C}{}^{C} - S_{;AB} - 2S_{(B}{}^{C}R_{AC)} + 2S^{CD}R_{ACBD} + 2\omega^{C}S_{C;(AB)} + 3\omega_{(A}S_{;B)} - 3\omega^{C}S_{AB;C} $$
$$ -2\omega_{(A}S_{B)C}{}^{;C} + 2S_{C(A}\omega_{B)}{}^{;C} - 2\omega^{C}{}_{;B}S_{AC} - \omega_A\omega_B S + \omega_C\omega^C S_{AB} = 0 $$

$R_{\dots} := {}^{(n)}R_{\dots}$ $\quad\quad$ $\cdot_{;A} := {}^{(n)}\nabla_A \cdot$ $\quad\quad$ $S := S_C{}^C$

Kolanowski, Lewandowski, Szereszewski 2019

Extremal Isolated Horizons to the 2nd order: equations

$$S_{AB;C}{}^{C} - S_{;AB} - 2S_{(B}{}^{C}R_{AC)} + 2S^{CD}R_{ACBD} + 2\omega^{C}S_{C;(AB)} + 3\omega_{(A}S_{;B)} - 3\omega^{C}S_{AB;C}$$
$$-2\omega_{(A}S_{B)C}{}^{;C} + 2S_{C(A}\omega_{B)}{}^{;C} - 2\omega^{C}{}_{;B}S_{AC} - \omega_{A}\omega_{B}S + \omega_{C}\omega^{C}S_{AB} = 0$$

Linear in S_{AB} on the background of ω_A, g_{AB} that satisfy the equation:

$$^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0$$

Remark: This equation is exact. However since it is linear, it also features in linear perturbations of Near Horizon Geometry spacetimes.

Lucietti, Li 2016

The Near Horizon Geometry spacetime

Given n **dimensional manifold** $\mathcal{S}$ **endowed with** g_{AB}, ω_A **such that**

$$ {}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0 $$

Define on $S \times \mathbb{R} \times \mathbb{R}$

$$ g_{\mu\nu}dx^\mu dx^\nu := g_{AB}dx^A dx^B - $$

$$ 2du\left[dv - 2v\omega_A dx^A - \frac{1}{2}v^2\left({}^{(n)}\nabla_A\omega^A + 2\omega^A\omega_A + \frac{2}{n}\Lambda\right)du\right] $$

Then

$$ {}^{(n+2)}G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 $$

and $H = S \times \mathbb{R} \times \{v = 0\}$ **is an extremal Killing horizon.**

$$ K = v\partial_v - u\partial_u, \qquad L = \partial_u $$

Pawłowski, Lewandowski, Jezierski 2004, Kunduri, J. Lucietti 2009, Real 2003, Kundt 1961

Near Horizon Geometry in 4-dim

The Near Horizon Geometry equation in 4-dim

$\mathcal{S}$ - a compact 2d-manifold equipped with:

$g_{AB}dx^A dx^B$ - a metric tensor, $\omega_A dx^A$ - a 1-form

$$^{(2)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$

K - the Gauss curvature

Λ - the cosmological constant

The type D equation as an integrability condition

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad d\omega =: \Omega \, i(\bar{m}_A m_B - \bar{m}_B m_A)$$

$$^{(2)}\nabla_{(A}\omega_{B)} + \omega_A \omega_B + \frac{1}{2}(\Lambda - K) g_{AB} = 0$$

$$\Downarrow$$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K - \frac{\Lambda}{3} + i\Omega \right)^{-\frac{1}{3}} = 0$$

DR, Lewandowski, Pawłowski 2018

Lemma. If S is compact, then everywhere

$$\Psi_2 = K - \frac{\Lambda}{3} + i\Omega \neq 0$$

DR, Kamiński, Lewandowski, Szereszewski 2019

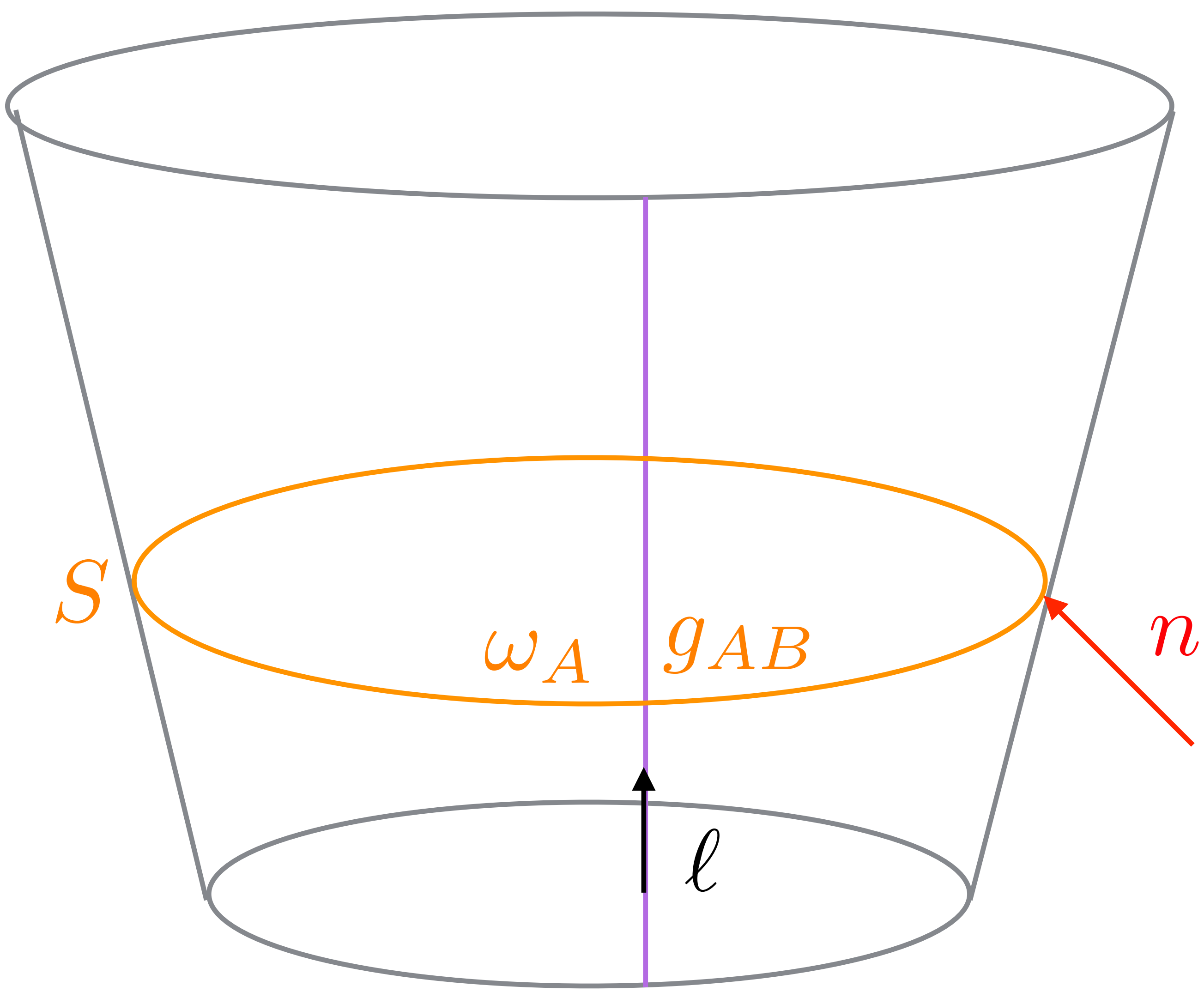
The emerging integrability condition is known on its own as the Petrov type D equation that applies to non-extremal isolated horizons.

Non-twisting
of the second principal null direction of the Weyl tensor

Theorem 7

Suppose (g_{AB}, ω_A) **satisfy the NHG equation:**

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$



then the null vector n **orthogonal to the corresponding slice** S **is:**

DDR, Lewandowski, Pawłowski (2018)

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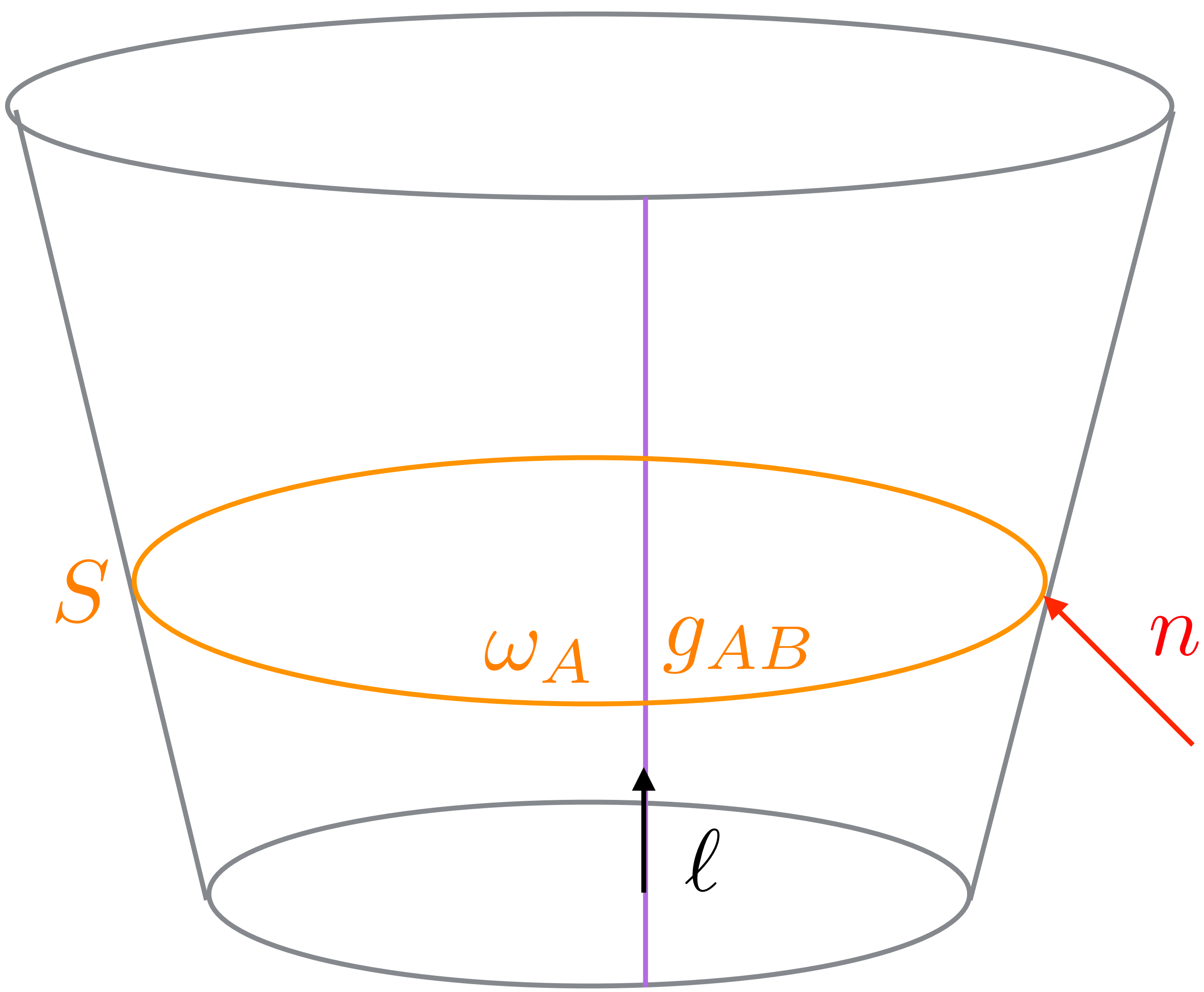

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1). non-expanding and shear-free

DDR, Lewandowski, Pawłowski (2018)

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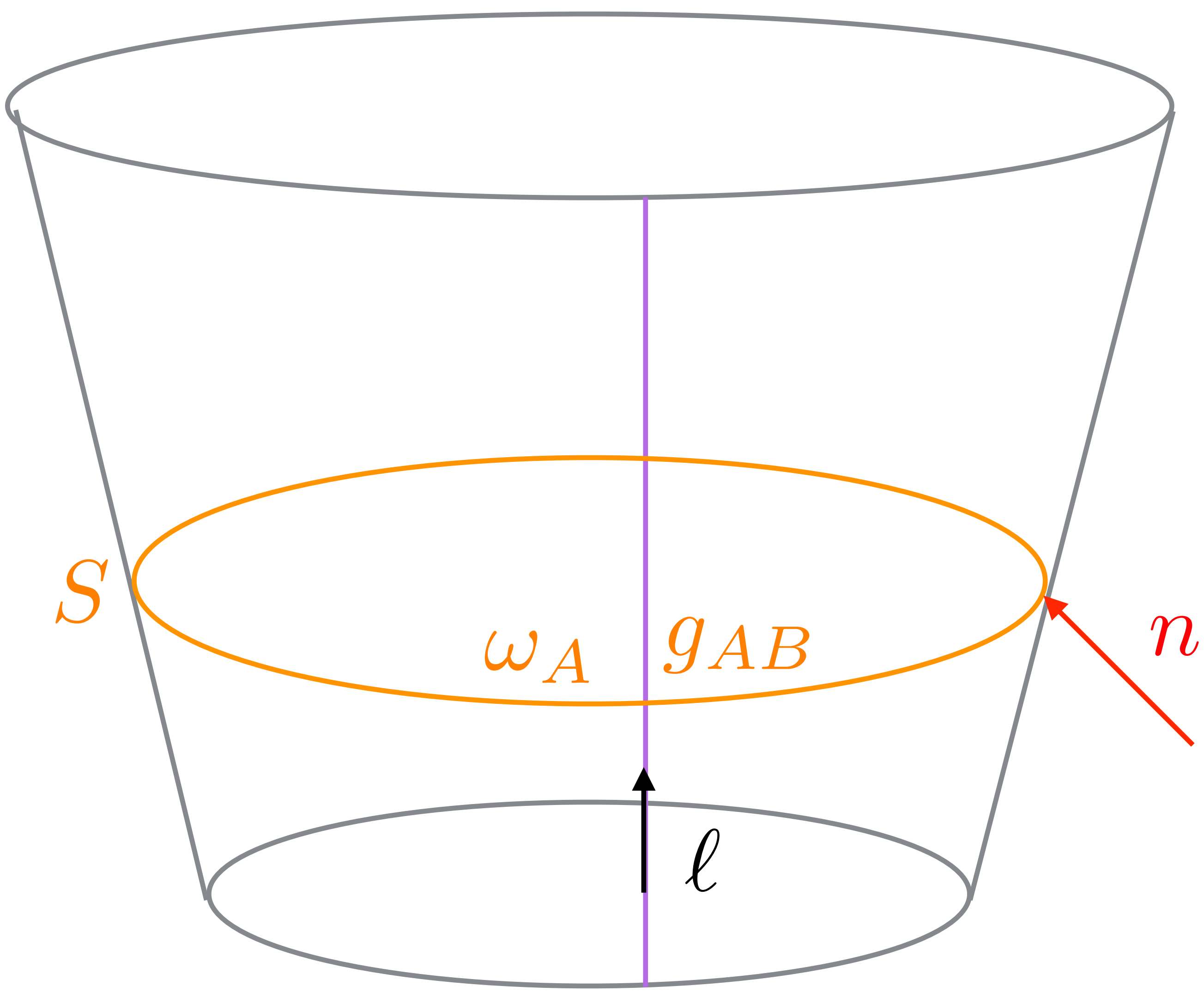


then the null vector n **orthogonal to the corresponding slice** S **is:**

1). non-expanding and shear-free

2). a double principal null direction

DDR, Lewandowski, Pawłowski (2018)

Extremal Isolated Horizons to the 2nd order: Uniqueness of the extremal Kerr horizon

Suppose $\mathcal{S} = S_2$ **and** g_{AB}, ω_A, S_{AB} **is axisymmetric and** $\Lambda = 0$

Then, the solution to the first and second NHG equation is unique, modulo the obvious rescaling:

$$g_{AB} \mapsto a\, g_{AB}, \qquad S_{AB} \mapsto b\, S_{AB}, \qquad a, b = \mathrm{const}$$

And it corresponds to the horizon in the extremal Kerr spacetime.

For every solution g_{AB}, ω_A, S_{AB} **the horizon** H, g_{ab}, ∇_a

is embeddable in the extremal Kerr spacetime of the corresponding horizon area.

Kolanowski, Lewandowski, Szereszewski 2019

Lewandowski, Pawłowski 2019 *Lucietti, Li 2016*

The Near Horizon Geometry equation in 4-dim: topological obstacles from the trace

$$g^{AB}| \quad {}^{(2)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$$

$$\Downarrow$$

$$\frac{4\pi}{A}(1 - \text{genus}) \ = \frac{1}{A}\int_S \omega_A\omega^A d\text{Area} \ + \Lambda \geq \ \Lambda$$

$$A := \int_S d\text{Area}$$

Allowed cases:

Pawłowski, L, Jezierski 2004,

Dobkowski-Ryłko, Kamiński, L, Szereszewski 2019

$$\text{genus} = 0, \ \Lambda \in \mathbb{R}$$
$$\text{genus} > 1, \ \Lambda < 0$$
$$\text{genus} = 1, \ \Lambda < 0$$
$$\text{genus} = 1, \ \Lambda = 0, \ \omega_A = 0 = K$$

NHG solutions for genus = 0

$$S = S_2$$

axial symmetry $\Lambda = 0$ $\Rightarrow$ $g_{AB},\ \omega_A = \ g_{AB}^{\mathrm{extremal\ \ Kerr}},\ \omega_A^{\mathrm{extremal\ \ Kerr}}$

Lewandowski, Pawłowski 2002,

generalized to the Einstein-Maxwell case

uniqueness! no more solutions!

generalized to the Einstein-Yang-Mills case
and somehow to the $\Lambda \neq 0$ **case** *Kunduri, J. Lucietti 2009*

no axial symmetry $\Rightarrow$ **?** **only partial results known:**

$${}^{(n)}\nabla_{[A}\omega_{B]} = 0 \qquad \Rightarrow K = \Lambda \geq 0,\ \omega_A = 0$$

(non-rotating) *Chruściel, Reall, Tod 2005*

the linearized equation about axisymmetric solution admits only axisymmetric solutions - partially numeric *Chruściel, Szybka, Tod 2017*

NHG solutions for genus > 0

$\mathcal{S}\,,\, g_{AB}\,,\, \omega_A$ $\qquad {}^{(2)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B + \frac{1}{2}(\Lambda - K)g_{AB} = 0$

K **- the Gauss curvature** $\qquad \Lambda$ **- the cosmological constant**

$$\chi_E({}^2\mathcal{S}) \le 0 \quad \Rightarrow \quad K = \Lambda \le 0, \qquad \omega_A = 0$$

(genus > 0)

Dobkowski-Ryłko, Kamiński, JL, Szereszewski 2018

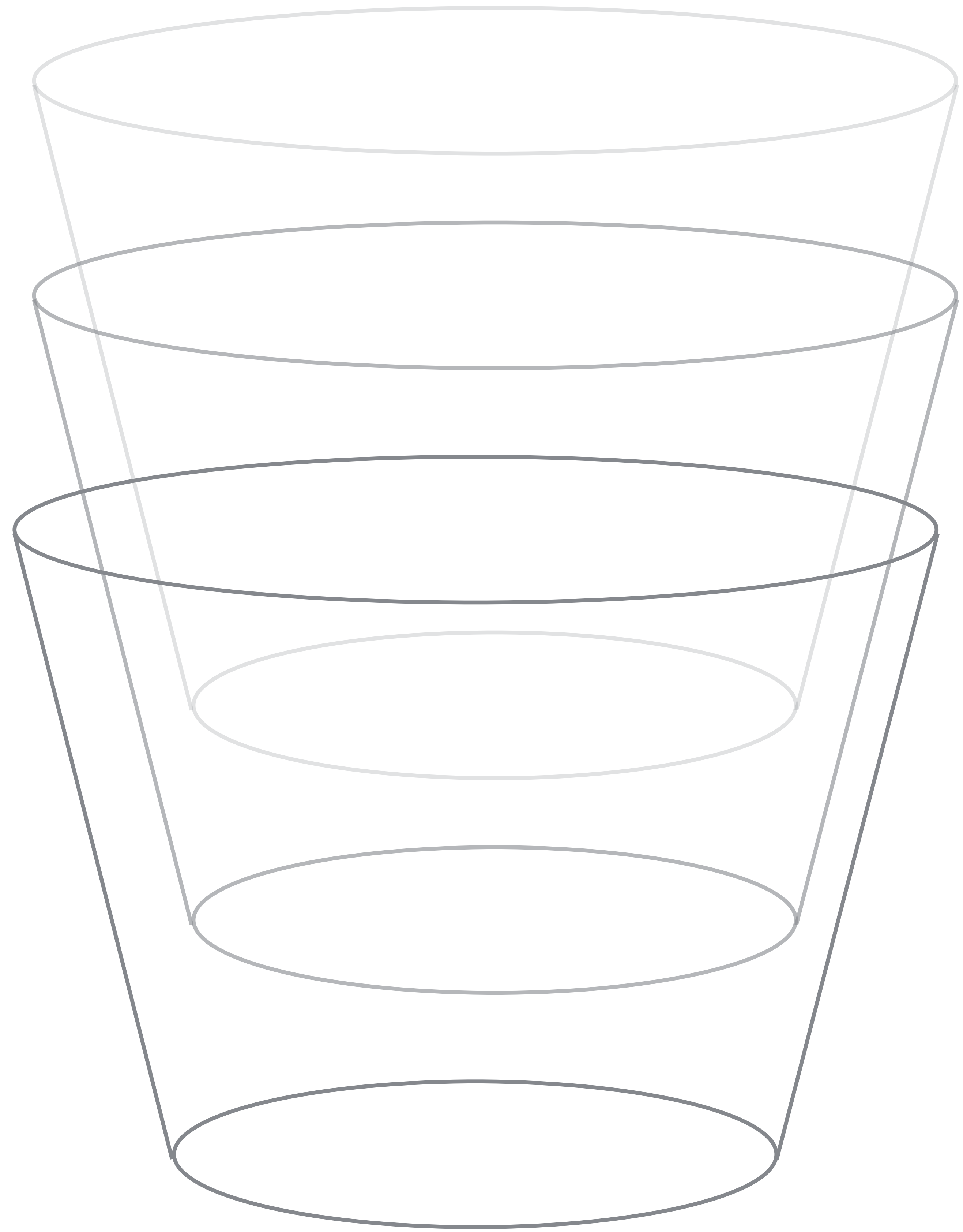
Embeddable in extremal cases $\Lambda = -\frac{1}{9M^2}$ **of:**

$$-(-1 - \frac{2M}{r} - r^2\frac{\Lambda}{3})dt^2 + \frac{dr^2}{-1 - \frac{2M}{r} - r^2\frac{\Lambda}{3}} + r^2\frac{2dzd\bar{z}}{(1 - \frac{1}{2}z\bar{z})^2}$$

this is really minus

compactified by suitable subgroup of isometries

Spacetimes foliated by non-expanding surfaces of co-dimension 1



Spacetimes foliated by non-expanding surfaces of co-dimension 1

Let $u = \text{const}$ **define the foliation. A distinguished choice for** ℓ

$$\ell = -g^{\mu\nu}\nabla_\nu u$$

On each space-like section of every non-expanding horizon the following foliation condition is satisfied:

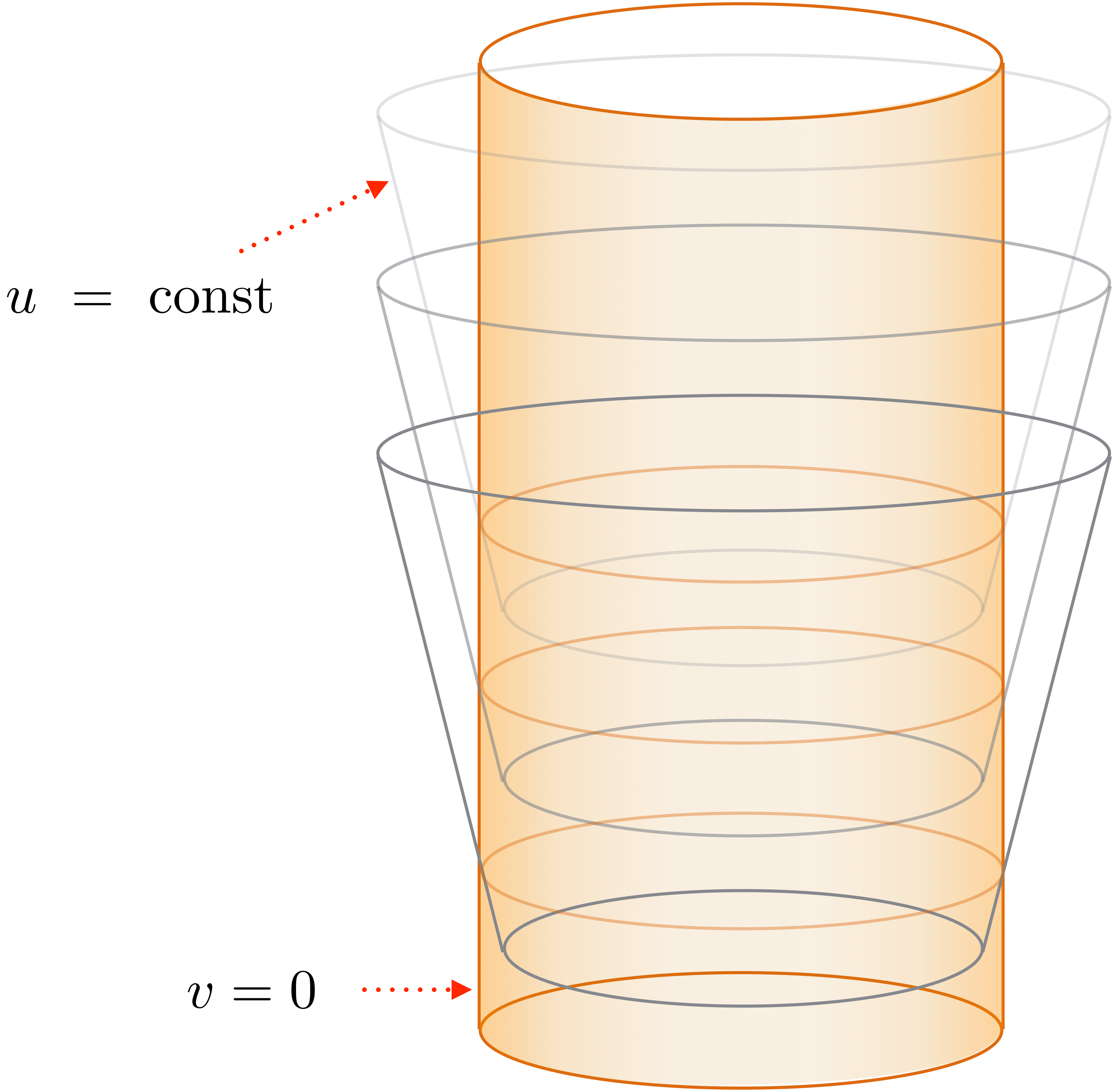
$$-{}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0$$

This equation is dual to the Near Horizon Geometry Equation, and each leaf of the foliation defines an abstract extremal IH geometry by $\omega_A \mapsto -\omega_A$ **.**

$${}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0$$

Kundt 1962, Podolsky 2008 L, Waluk, Szereszewski 2016, L, Szereszewski 2018
Pawłowski, L, Jezierski 2004,

Non-expanding horizon foliation and a transversal horizon



Spacetimes foliated by non-expanding horizons

Theorem 8. Suppose 4-dim spacetime $M, g_{\mu\nu}$ is foliated by non-expanding horizons emanating from a single transversal isolated horizon; then the vacuum Einstein equations with a cosmological constant Λ are satisfied if and only if this is a near horizon geometry, namely

$$g_{\mu\nu}dx^\mu dx^\nu \quad := \quad g_{AB}dx^A dx^B -$$

$$2du\left[dv - 2v\omega_A dx^A - \frac{1}{2}v^2\left({}^{(n)}\nabla_A\omega^A + 2\omega^A\omega_A + \frac{2}{n}\Lambda\right)du\right]$$

and

$$-{}^{(n)}\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}{}^{(n)}R_{AB} + \frac{1}{n}\Lambda g_{AB} = 0$$

Lewandowski, Szereszewski, Waluk 2016, Lewandowski, Szereszewski 2018

Summary

- The type D equation: $\bar{m}^A \bar{m}^B \nabla_A \nabla_B \Psi_2^{-\frac{1}{3}} = 0$
- Non-twisting of the second double principal vector if: $\nabla_{(A}\omega_{B)} + \omega_A \omega_B + \frac{1}{2}(\Lambda - K) g_{AB} = 0$
- All axisymmetric solutions to the type D eq. for trivial p.f.b. structure on topological 2-sphere parametrized by (A, J);
- All solutions on genus>0 derived (non-rotating);
- All axisymmetric solutions solutions to the type D eq. for a non-trivial p.f.b structure on a 2-sphere derived, generic 3-parameter solution non-embeddable in the known generalized black hole solutions;
- Open problem: existence of non-axisymmetric solutions on topological sphere;
- Bifurcated horizon - axial symmetry without rigidity theorem;
- Type D equation as an integrability condition for NHG;
- NHG solutions for genus=0 and genus>0;