Renormalisation in LQC with SU(1,1) techniques

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based on:
1811.02792 (w. Fabian Haneder) and 1904.13269 (w. Dennis Wuhrer)

Jurekfest, September 20, 2019

Talk in a nutshell

Coarse graining with SU$(1,1)$ techniques:

- **Why is it interesting?**
  - Renormalisation necessary in QFT
  - Largely ignored in LQG / LQC (complicated!)
  - Analytically tractable models are welcome
- **What do we do?**
  - Coarse graining for cosmological spacetimes
  - Technical basis: su$(1,1)$ algebra
  - Everything works on (copies of) LQC Hilbert space
- **What is new?**
  - Analytic "many small" to "few large" spin coarse graining
  - Renormalised large spin Hamiltonian capturing small spin physics
  - Example of error made when neglecting renormalisation

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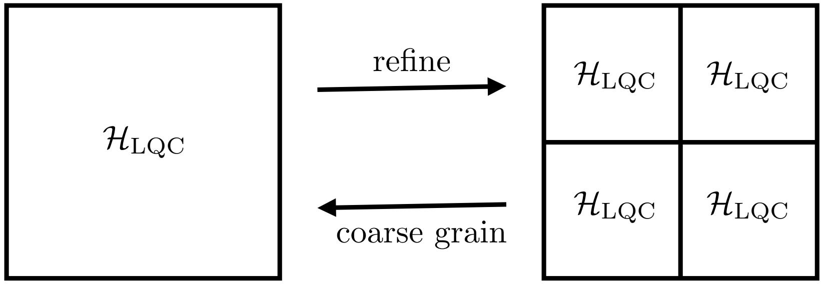
Table of contents

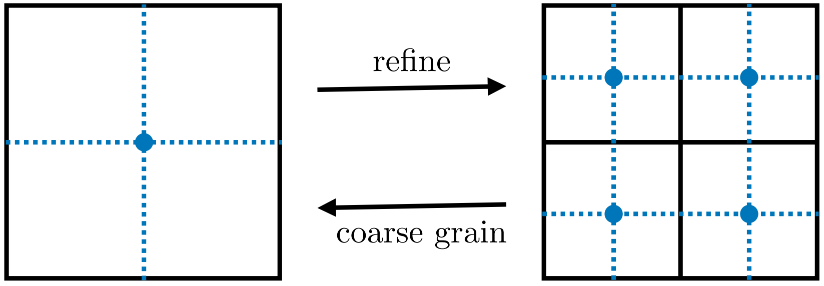
Outline

Coarse graining in LQC



- N copies of LQC, each in one cell
- States are the same in all cells (homogeneity)
- Coarse graining via adding extensive quantities: $\langle \hat{v} \rangle = \sum_{i=1}^{4} \langle \hat{v}_i \rangle$
- No interactions between cells (fixed # cells + interactions: [Wilson-Ewing '12])

Coarse graining in a full theory embedding of LQC



- N vertices, each associated with one cell
- States are the same at all vertices (homogeneity, similar to GFT condensate)
- Coarse graining via adding extensive quantities: $\langle \hat{v} \rangle = \sum_{i=1}^{4} \langle \hat{v}_i \rangle$
- No interactions between cells (inhomogeneous operators are weakly zero)
- LQC-type variables: $\bullet = \mathcal{H}_{\mathrm{LQC}}$ [NB '15, '16 (Warsaw years)]

Outline

Setup of SU(1,1) group quantisation techniques

[LQC application: Dupuis, Livine '12; Ben Achour, Livine '17-'19]

SU(1,1) reminder

- (Poisson-)Algebra: $\{k_x, k_y\} = -j_z, \quad \{k_y, j_z\} = k_x, \quad \{j_z, k_x\} = k_y$
- Representations: $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ quantum numbers: $m = j, j+1, j+2, \ldots$

Requirements

- Poisson-algebra of observables to be quantised isomorphic to su(1,1)
- Dynamics generated by linear combinations of algebra generators
- All 3 observables are extensive, e.g. CVH algebra

Basic strategy

- Quantisation as su(1,1) generators in representation j
- $m \propto$ volume, $\quad j =$ lower volume cutoff $=$ RG-scale

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Coherent states

Operators and Hilbert space: representation $j \in \mathbb{N}/2$, q-number $m \in j, j+1, \ldots$

$$\hat{j}_z \, |j, m\rangle = m \, |j, m\rangle \,, \qquad \hat{k}_\pm \, |j, m\rangle \propto |j, m \pm 1\rangle \,, \qquad k_\pm = k_x \pm i k_y$$

Perelomov coherent states

$$|j, z\rangle \propto \sum_{m=j}^{\infty} \sqrt{\binom{m+j-1}{m-j}} \frac{(z^1)^{m-j}}{(\bar{z}^0)^{m+j}} \, |j, m\rangle \,, \qquad z^0, z^1 \in \mathbb{C}^2$$

Expectation values

$$\langle j, z \,|\, \hat{g} \,|\, j, z\rangle = j \cdot f_g(z), \qquad \hat{g} = c_z \hat{j}_z + c_\pm \hat{k}_\pm$$

→ Suggests coarse graining operation: $j \mapsto Nj$ extensive scale

$z \mapsto z$ intensive state

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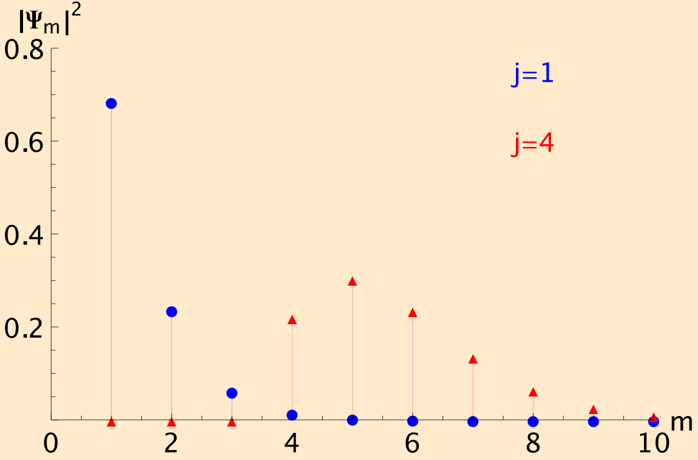
Example coarse graining: $j = 1 \mapsto j = 4$



Higher moments and matrix elements

Coarse graining exact eigenvalues and their probabilities

$$N \times (j + \mathbb{N}_0) = N \times j + \mathbb{N}_0, \qquad P_{Nj}(m) = \sum_{\sum m_i = m} \prod_i P_j(m_i)$$

Coarse graining exact for all higher moments

$$\begin{aligned}
\left\langle \hat{j}_\alpha^n \right\rangle_j &= \left\langle \left(\hat{j}_{\alpha,1} + \ldots + \hat{j}_{\alpha,N} \right)^n \right\rangle_{j_0,1,\ldots,N} \\
&= \sum_{\substack{r_1,\ldots,r_j=0: \\ n=r_1+\ldots+r_j}}^{n} \frac{n!}{r_1! r_2! \ldots r_j!} \left\langle \hat{j}_\alpha^{r_1} \right\rangle_{j_0} \cdots \left\langle \hat{j}_\alpha^{r_j} \right\rangle_{j_0} \\
&= \sum_{m=1}^{n} \frac{N!}{(N-m)!} \sum_{\substack{1 \le k_1 \le \ldots \le k_m: \\ n=k_1+\ldots+k_m}} \frac{n!}{k_1! k_2! \ldots k_m!} \prod_{p=1}^{n} \frac{1}{(\#k_i = p)!} \left\langle \hat{j}_\alpha^p \right\rangle_{j_0}^{(\#k_i=p)}
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Dynamics and coarse graining

Coherent state property

$$U\,|j,z\rangle = |j, U\cdot z\rangle \quad \forall \quad U \in \mathrm{SU}(1,1)$$

Dynamics generated by su$(1,1)$ element $\Rightarrow$ acts only on intensive state

Dynamics is j-independent $\Rightarrow$ commutes with coarse graining

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Strategy

Up to now:

- Quantise classical su$(1,1)$ Poisson-algebra
- su(1,1) representations not in terms of fundamental LQC operators

Goal:

- Implement on N copies of $\mathcal{H}_{\mathrm{LQC}}$
- Start with $\hat{v}$ and $\widehat{e^{in\lambda b}}$ as operators, $\quad n \in \mathbb{Z}, \quad \lambda \propto \beta^{3/2}$
- Find 1-parameter family of operators reproducing su$(1,1)$ algebra (rep. j)
  [Null representation: Livine, Ben Achour '19]
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su(1,1) from LQC operators

Regularised CVH algebra:

$$\hat{j}_z = \frac{\hat{v}}{2\lambda}$$ volume eigenvalues: $2\lambda m$

$$\hat{H}_g = -\frac{1}{2\lambda^2}\widehat{\sin(\lambda b)}\sqrt{\hat{v}^2 - \tilde{v}_m^2}\widehat{\sin(\lambda b)}$$ suggested by minimal volume

$$\hat{C} = -i\left[\hat{v}, \hat{H}_g\right] = \frac{1}{2\lambda}\left[\widehat{\cos(\lambda b)}\sqrt{\hat{v}^2 - \tilde{v}_m^2}\widehat{\sin(\lambda b)} + \widehat{\sin(\lambda b)}\sqrt{\hat{v}^2 - \tilde{v}_m^2}\widehat{\cos(\lambda b)}\right]$$

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$$\hat{k}_x = \frac{1}{2\lambda}\left[4\lambda^2\hat{H}_g + \frac{1}{2}\left(\sqrt{\ldots} + \sqrt{\ldots}\right)\right], \qquad \hat{k}_y = \hat{C}, \qquad \hat{j}_z = \frac{1}{2\lambda}\hat{v}$$

Representation via Casimir operator:

$$\hat{C}\,|2\lambda m\rangle \stackrel{!}{=} j(j-1)\,|2\lambda m\rangle \qquad \Rightarrow \tilde{v}_m = 2\lambda\left(j - \frac{1}{2}\right)$$

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Renormalised Hamiltonian

Reminder

- Action of algebra generators commutes with coarse graining
- $H_g = \frac{1}{2\lambda}(k_x - j_z)$ classically

Renormalised Hamiltonian:

$$\hat{H}_g^{\text{renormalised}}(j) = \frac{1}{2\lambda}(\hat{k}_x - \hat{j}_z)$$

$$= -\frac{1}{2\lambda^2}\widehat{\sin(\lambda b)}\,\sqrt{\hat{v}^2 - 4\lambda^2(j-1/2)^2}\,\widehat{\sin(\lambda b)}$$

$$+\frac{1}{8\lambda^2}\left(\sqrt{(\hat{v}+\lambda)^2 - 4\lambda^2(j-1/2)^2} + \sqrt{(\hat{v}-\lambda)^2 - 4\lambda^2(j-1/2)^2} - 2\hat{v}\right)$$

- No corrections for $j = 1/2$
- Small volume corrections for $j > 1/2$
- Hamiltonian preserves $v \in (2j + \mathbb{N}_0)\lambda$ subspaces

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Bounce-volume dependence of the critical density I

- Non-graph changing: volume at bounce time = renormalisation scale
- All Perelomov coherent states evolved with $\hat{H}_g^{\text{ren.}}(j)$, e.g. $j = \frac{1}{2}$

$$\left\langle \hat{H}_g^{\text{ren.}} \right\rangle = -\frac{j}{2\lambda}\frac{1}{\alpha}, \qquad \langle v(t_\text{b})\rangle = \lambda j \frac{1+\alpha^2}{\alpha} \geq 2\lambda j$$

$$\rho_\text{b} = \frac{\left\langle \hat{H}_g^{\text{ren.}} \right\rangle}{\langle v(t_\text{b})\rangle} = \frac{1}{2\lambda^2}\frac{1}{1+\alpha^2} \leq \frac{1}{2\lambda^2}$$

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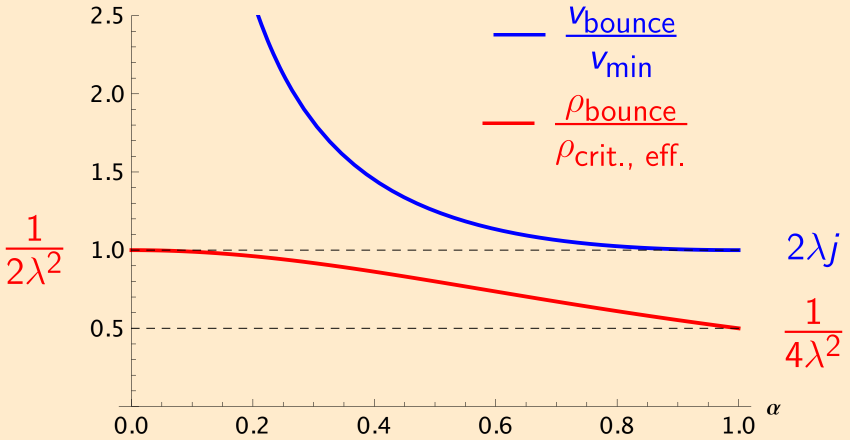

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- **Take-away lessons**
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  - Possibility of analytic treatment is surprising
  - Low spin physics may be different from effective equations
- **Outlook:**
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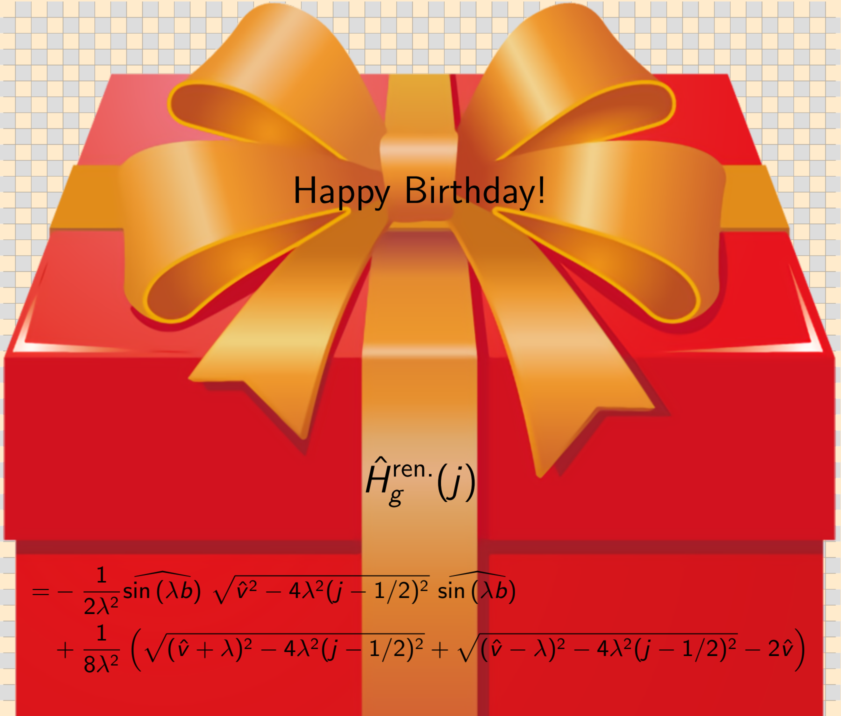

Happy Birthday!

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