Lie series method in a perturbed black hole spacetime

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Introduction

We study the motion of test particles around Schwarzschild black hole perturbed by a ring-like source. In order to solve the equations of motion we employ the Lie series formalism which enables us to approximately transform our hamiltonian into action-angle coordinates i.e. to compute the so-called Birkhoff normal form of hamiltonian. Finding this transformation is then effectively equivalent to solving the Hamilton equations. The results are then confronted with the solution obtained by numerical integration. In the future we plan to use this approximation in an astrophysical setting.

Action-angle coordinates and Lie transform

Bounded orbits of an integrable system can be parametrized by the action angle coordinates in which the hamiltonian takes a simple form:

$$(q_i, p_i) \to (\psi_i, J_i), \quad H(q_i, p_i) \to H(J_i).$$

In these coordinates the Hamilton equations can be trivially solved

$$J_i = \text{constant}, \quad \psi_i = \Omega_i t + \psi_{0i}.$$

This can be partially extended to nearly integrable systems in the form of an expansion in a small parameter ε:

$$H^{(0)} = H_0(J_i) + \sum_{i=1} \varepsilon^i H_i^{(0)}(\psi_i, J_i)$$

using the canonical transformations defined by the action of the Lie operator:

$$H^{(1)} = \exp(\varepsilon \pounds_{\omega_1}) H^{(0)}, \quad \pounds_g f = \{f, g\}$$

where the Lie derivative is represented by the Poisson bracket while the generating function χ is determined by the homological equation

$$\{H_0, \omega_1\} + h_1 \overset{!}{=} 0$$

where h_1 is the part of $H_1^{(0)}$ depending on angles. The coordinate transformations are also determined by the Lie operator:

$$z_i^{old} = \exp(\varepsilon \pounds_{\omega_1}) z_i^{new.}$$

Successive application of the Lie operators then results in elimination of all terms containing angles up to the desired order thus obtaining the Birkhoff normal form of hamiltonian which depends only on actions while the higher order terms are neglected. As already mentioned above the equations of motion in these coordinates can be easily solved. This solution can then be inserted into the transformation relations between the old and the new coordinates yielding an approximate solution to the original problem.

The metric

The studied spacetime belongs to the static and axially symmetric class of solutions of Einstein equations described by the Weyl metric

$$\mathrm{d}s^2 = -e^{2\nu(\rho,z)}\mathrm{d}t^2 + e^{2\lambda(\rho,z)-2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2 e^{-2\nu(\rho,z)}\mathrm{d}\phi^2.$$

fully described by two metric functions ν and λ which satisfy the field equations:

$$\nu_{,\rho\rho} + \frac{1}{\rho}\nu_{,\rho} + \nu_{,zz} = 0, \quad \lambda_{,\rho} = \rho\left[(\nu_{,\rho})^2 + (\nu_{,z})^2\right], \quad \lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z}.$$

The first equation is linear and therefore we can easily superpose the Schwarzschild black hole and a ring with masses M and m respectively. The Schwarzschild metric functions read:

$$\nu_{Schw}(\rho, z) = \frac{1}{2}\ln\left(\frac{d_1 + d_2 - 2M}{d_1 + d_2 + 2M}\right), \quad \lambda_{Schw}(\rho, z) = \frac{1}{2}\ln\left(\frac{(d_1 + d_2)^2 - 4M^2}{4d_1 d_2}\right)$$

where $d_{1,2} = \sqrt{\rho^2 + (z \mp M)^2}$. The ring to be used is the Bach-Weyl ring however the second metric function of such a superposition would be rather complicated and so we instead assume that the ring is sufficiently distant from the black hole which means that the asymptotic expansion in the ring radius b can be performed. The first non-trivial term of the expansion then corresponds to the quadrupole:

$$\nu_{ring} = -\frac{1}{4}\frac{m}{b^3}\left(\rho^2 - 2z^2\right), \qquad \lambda_{ring} = \frac{1}{16}\frac{m^2}{b^6}\rho^2\left(\rho^2 - 8z^2\right).$$

The total superposition is then

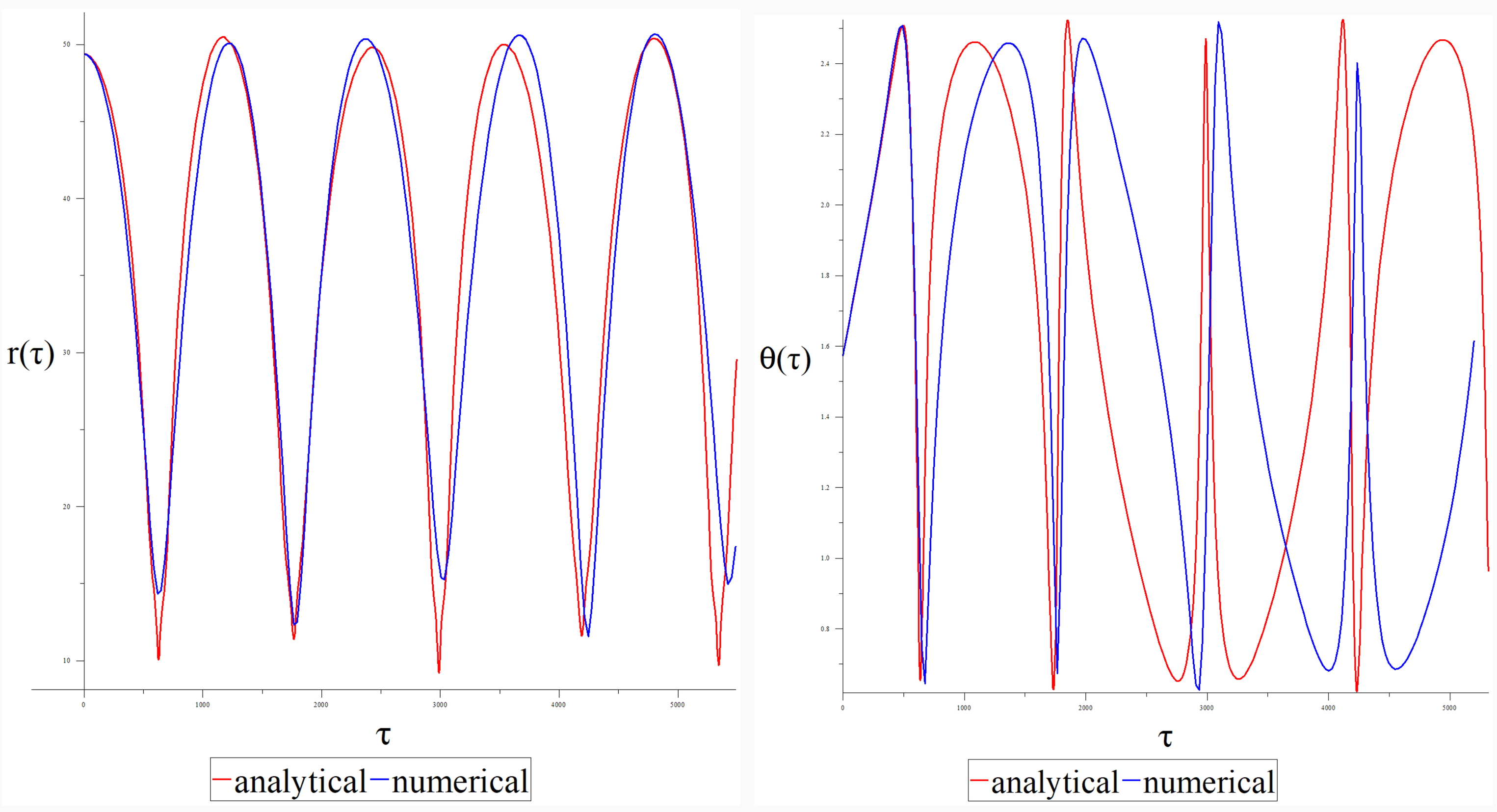
$$\nu = \nu_{Schw} + \nu_{ring}, \quad \lambda = \lambda_{Schw} + \lambda_{ring} + \lambda_{cross}$$

where the second metric function λ includes the cross term

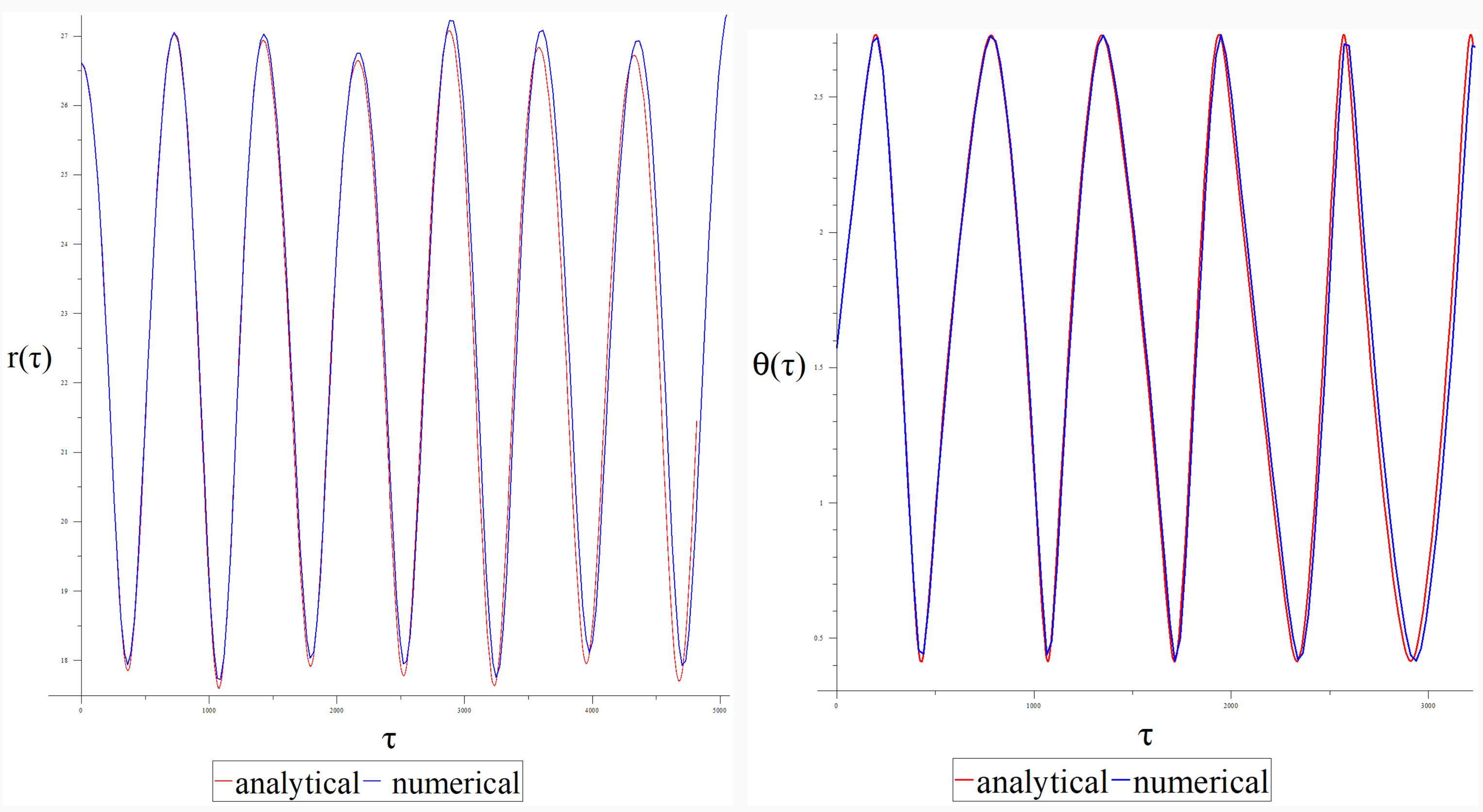
$$\lambda_{cross} = -\frac{1}{2}\left((z + M)\, d_1 + (M - z)\, d_2\right)\frac{m}{b^3}.$$

Comparison with numerical results

orbit 1



orbit 2



Two bound orbits described by spherical coordinates r and θ as functions of proper time τ. The geodesics obtained analytically using the Lie series method are compared to the numerical solution of the geodesic equation.

The analytical approximation

Unlike its Newtonian counterpart the Schwarschild hamiltonian cannot be exactly transformed into action-angle coordinates therefore we shall for simplicity concentrate only on the low eccentric orbits. The Schwarzschild hamiltonian reads:

$$H_{Schw} = \frac{1}{2}\Big[-\frac{1}{1 - \frac{2M}{r}}p_t^2 + \left(1 - \frac{2M}{r}\right)p_r^2 + \frac{1}{r^2}\Big(p_\theta^2 + \frac{J_\phi^2}{\sin^2\theta}\Big)\Big],$$

In order to separate the radial and angular variables we replace the evolution parameter τ with λ: $\mathrm{d}\tau = r^2\mathrm{d}\lambda$. The angular part can be solved trivially using the transformation

$$J_\theta = L - J_\phi, \quad \theta = \pi - \arccos\left(\sqrt{1 - \frac{{J_\phi}^2}{(J_\phi + J_\theta)^2}}\sin(\psi_\theta)\right)$$

where L is the total angular momentum. The hamiltonian then takes form:

$$H_{Schw} = H_{rad}(r, p_r) + \frac{1}{2}(J_\theta + J_\phi)^2$$

The radial part H_{rad} can be worked out as a series in eccentricity ε:

$$H_{rad} = H_0 + J_r\Omega + \mathcal{O}(\varepsilon^3)$$

where the second term is just a harmonic oscillator with action J_r measuring the distance from the stable circular orbit. Applying twice the Lie operator gives us the normal form of hamiltonian with a negligible remainder:

$$\exp(\pounds_{\omega_2})\exp(\pounds_{\omega_1})H_{Schw} = H_{NF}(J_r, J_\theta) + \mathcal{O}(\varepsilon^5)$$

The total hamiltonian is first linearised in the perturbation parameter $Q = \frac{m}{b^3}$:

$$H_{tot} = H_{Schw} + \frac{\partial H_{tot}}{\partial Q}Q + \mathcal{O}(Q^2)$$

and then by successive actions of the operators $\exp(\pounds_{\omega_1})$, $\exp(\pounds_{\omega_2})$ and $\exp(Q\pounds_\chi)$ approximately transformed into the Birkhoff normal form

$$H_{tot} = H_{NF}(J_r, J_\theta) + QZ_{Q1}(J_r, J_\theta) + \mathcal{O}(Q^2).$$

The limits of validity of our approximation are given by our assumptions i. e. that the parameter Q and eccentricity ε are sufficiently small. This can be clearly seen in the figures on the left where the more eccentric analytical orbit slightly deviates from the numerical solution.

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