Schwinger-Dyson equations of Tensor Field Theory as Tutte-like equations in dimension 3

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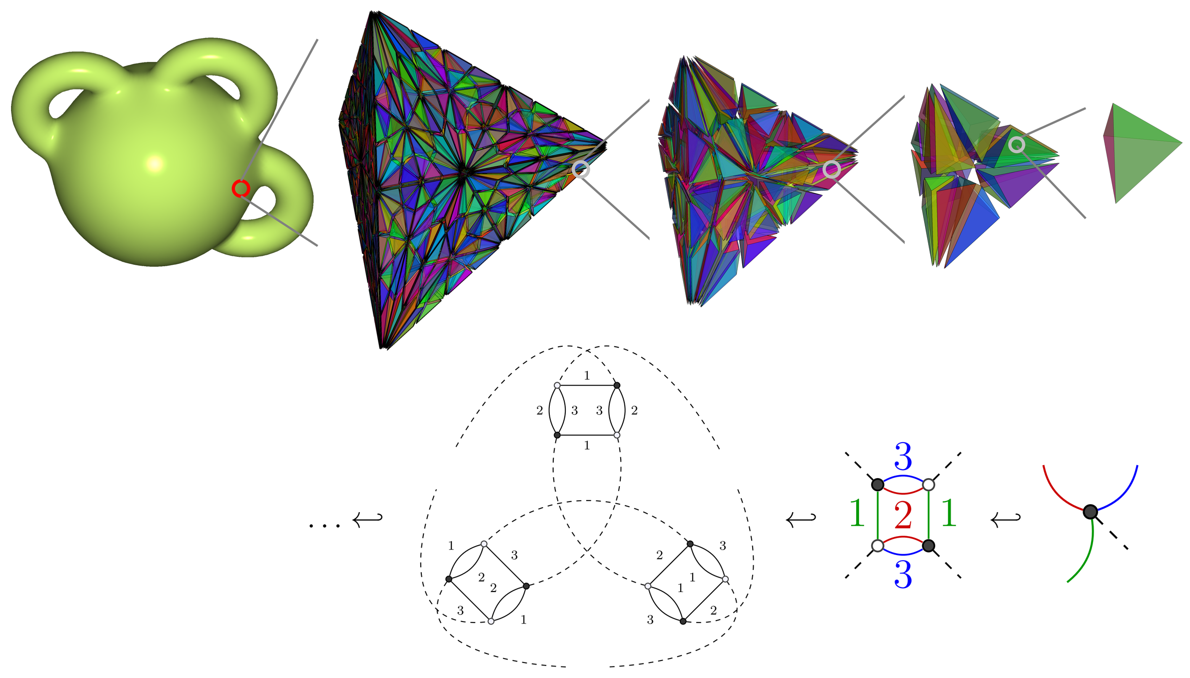
Motivation: A theory of random discrete (graph-encoded) spacetime

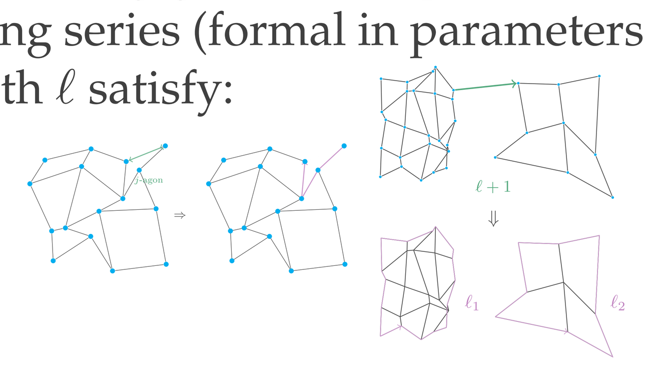


Aim: Non-perturbative techniques in Tensor Models, analytic Schwinger-Dyson equations for tensor models. N.B. Upper row figures by Wikipedia (Oleg Alexandrov). Barycentric subdivision by Mathematica-code by Dror Bar-Natan.

Tutte's Equations for maps and matrix models

Here a *map* is a discrete oriented surface, possibly with boundary. Tutte's equations [Tut63] relate the generating series of maps of neighboring genera and perimeters (i.e. number of edges) of the boundary(ies). Generating series (formal in parameters λ_α) of planar maps $\mathcal{T}_\ell$ with a single boundary of length ℓ satisfy:

$$\text{(planar)}\quad \mathcal{T}_{\ell+1} = \sum_\alpha \lambda_\alpha \mathcal{T}_{\ell-1+\alpha} + \sum_{\ell_1+\ell_2=\ell-1} \mathcal{T}_{\ell_1}\times\mathcal{T}_{\ell_2}$$



In general, maps can be drawn on surfaces of genus g and boundaries of lengths $K=\{\ell_1,\ldots,\ell_k\}$; the corresponding generating series is $\mathcal{T}_K^{(g)}$. Generally

$$\mathcal{T}^{(g)}_{l_0+1,K} = \sum_{\alpha=3}^{d}\lambda_\alpha \mathcal{T}^{(g)}_{l_0+\alpha-1,K} + \sum_{m=1}^{k} l_m \mathcal{T}^{(g)}_{l_0+l_m-1,K\setminus\{l_m\}} + \sum_{j=0}^{l_0-1}\Big[\mathcal{T}^{(g-1)}_{j,l_0-1-j,K} + \sum_{\substack{g_1+g_2=g\\ J\subset K}} \mathcal{T}^{(g_1)}_{j,J}\times\mathcal{T}^{(g_2)}_{l_0-1-j,K\setminus J}\Big].$$

These are the Schwinger-Dyson equations (SDE) [Mig83] of a suitable matrix model.

What about higher dimensions? Tensor field theory ...

- ...is a field theory for tensors $\varphi,\bar\varphi:\mathcal{H}_1\otimes\mathcal{H}_2\otimes\ldots\otimes\mathcal{H}_D\longrightarrow\mathbb{C}$ independently transforming under each unitarity $W^{(k)}\in\mathcal{U}(\mathcal{H}_k)$ in the fundamental representation:

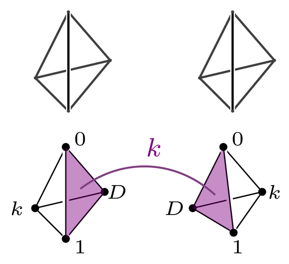
$$\varphi_{a_1a_2\ldots a_D}\longmapsto\varphi'_{a_1a_2\ldots a_D}=\textstyle\sum_{b_k}W^{(k)}_{a_kb_k}\varphi_{a_1a_2\ldots b_k\ldots a_D},\quad \bar\varphi_{p_1p_2\ldots p_D}\longmapsto\bar\varphi'_{p_1p_2\ldots p_D}=\sum_{q_k}\overline{W}^{(k)}_{p_kq_k}\bar\varphi_{p_1p_2\ldots q_k\ldots p_D}.$$

- There is a bijection $\{$unitary invariants$\}\leftrightarrow\{D$-colored graphs$\}$. E.g. for $D=3$

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{p},\mathbf{q}}(\bar\varphi_{q_1q_2q_3}\bar\varphi_{p_1p_2p_3})(\delta_{a_1p_1}\delta_{a_2q_2}\delta_{a_3q_3}\delta_{b_1q_1}\delta_{b_2p_2}\delta_{b_3p_3})(\varphi_{a_1a_2a_3}\varphi_{b_1b_2b_3})\leftrightarrow 1\square 1$$

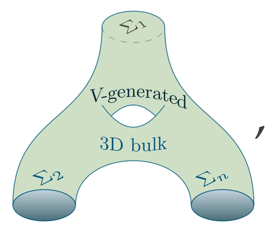
One constructs models S from finite sums of these invariants. Feynman diagrams turn out to be $D+1$ colored graphs — the extra color being a propagator— known to be a dual description of PL-(pseudo)manifolds, possibly with boundary:

Field Theory		Graph Theory		Topology & Geometry
φ $\quad\bar\varphi$	$\longrightarrow$	$\circ$ $\quad\bullet$	$\longrightarrow$	
$\varphi_{\ldots i_k\ldots}\delta_{i_kj_k}\bar\varphi_{\ldots j_k\ldots}$	$\longrightarrow$	$\circ\overset{k}{\frown}\bullet$	$\longrightarrow$	



- $\log\mathcal{Z}[J,\bar J] = \log\int\mathcal{D}\varphi\mathcal{D}\bar\varphi\, e^{\mathrm{Tr}(\bar J\varphi)+\mathrm{Tr}(\bar\varphi J)-N^{D-1}S(\varphi,\bar\varphi)} = \sum_{\mathcal{B}\text{ boundary}}\frac{1}{\sigma(\mathcal{B})}G_{\mathcal{B}}\star\mathbb{J}(\mathcal{B})$.

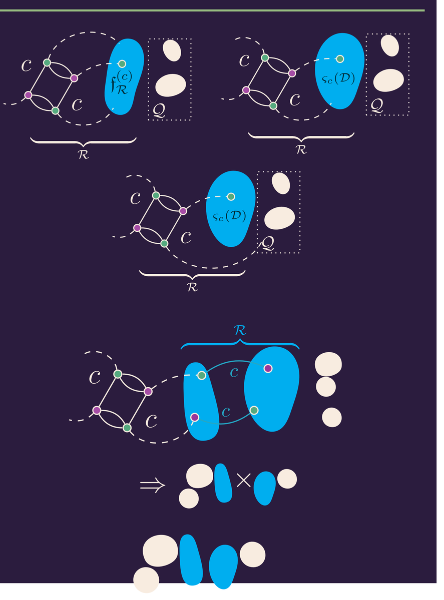
  Notation: $\mathbb{J}(1\square 1)(\mathbf{x},\mathbf{y}) = J_{\mathbf{x}}\bar J_{y_1x_2x_3}J_{\mathbf{y}}\bar J_{x_1y_2y_3}$ and the star $\star$ sums over indices.

- Geometric picture of the correlators $G_{\mathcal{B}_1|\mathcal{B}_2|\ldots|\mathcal{B}_n}\longleftrightarrow$  , $\mathcal{B}_\alpha\longleftrightarrow\Sigma_\alpha$, $D=3$.

Results [Pér18a]

Tutte's equations can be stated in terms of operations on the set of perimeters of the boundary components. The analogous SDE's of a (tensor field theory) correlator $G_{\mathcal{D}}$ turn out to be given by operations on a colored graph $\mathcal{D}$ that dually triangulates the boundary. For an arbitrary disconnected graph $\mathcal{D}=\mathcal{R}_{\text{conn.}}\sqcup\mathcal{Q}$, just as an edge was marked in Tutte's derivation, $\mathcal{R}_{\text{conn.}}$ has a marked vertex (dual to a marked triangle). Although the operations on the input graph $\mathcal{D}$ are more involved than in the 2-dimensional case, they all match operations appearing in Tutte's equations. For the model $S[\varphi,\bar\varphi]=\langle\bar\varphi,E\varphi\rangle+\lambda(1\square1+2\square2+3\square3)$ these are:

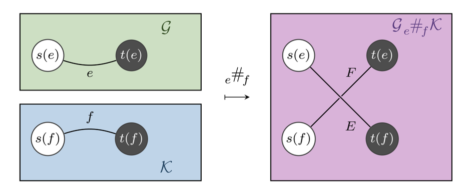
Framework→ Description ↓	Tutte's Equations disconnected ∂	SDE of TFT disconnected ∂
Merge with interaction	l_0-1+j	
Merge two boundaries	$l_0-1+l_\alpha, K\setminus\{l_\alpha\}$	
Split boundaries:		
Case I. # preserved	$\underbrace{(i,I)\times(m,M)}_{i+m=l_0-1\ \&\ I\cup M=K}$	
Case II. # increases	i,m,K	



Tools

- *Completeness of boundary geometries.*

  QFT-compatible connected sum # on graphs [Pér17]. The operation # is well-defined (binary) on Feynman graphs of any model and satisfies $\omega(\mathcal{G}\#\mathcal{K})=\omega(\mathcal{G})+\omega(\mathcal{K})$.



  Here ω is Gurău's degree, the tensor analogue of the genus in 't Hooft's $1/N$-expansion. This operation is used to show that the quadratic model with interaction $\lambda(1\square1+2\square2+3\square3)$ generates all boundary geometries [Pér18b].

- *The Ward-Takahashi Identity* [Pér18b] The unitary symmetries yield this identity, which can be used to trade certain derivatives $\delta^2\mathcal{Z}/\delta J\delta\bar J$ by $(J\cdot\delta\mathcal{Z}/\delta J-\bar J\cdot\delta\mathcal{Z}/\delta\bar J)$. This helps to descend the pyramid of correlators:

$n=8$ $\quad\ddots\quad\vdots\quad⋰$
SDE ⟲ WTI
$n=6$ $\quad G_{\square 1}\ G_{\square 2}\ G_{\square 3}\ G\ G_{2\,3}\ G_{1\,3}\ G_{1\,2}\ \cdots$
SDE ⟲ WTI
$n=4$ $\quad G_{1\square1}\ G_{2\square2}\ G_{3\square3}\ G_{\ominus|\ominus}$
SDE ⟲ WTI
$n=2$ $\quad G_{\ominus}$

- *Connected-boundary Schwinger-Dyson pyramid* (in collaboration with R. Pascalie and R. Wulkenhaar [PPW17]). The Ward-Takahashi identity can be inserted to compute a correlator as certain connected graph-derivatives (see below) on $\mathcal{Z}[J,\bar J] = e^{-\lambda(1\square1+2\square2+3\square3)}|_{\text{sources}}(\mathcal{Z}_{\text{free}}[J,\bar J])$. For disconnected-boundary correlators, one needs the graph calculus.

- *Graph calculus.* The free energy $\log\mathcal{Z}$ and other interesting functionals U are spanned by colored graphs (that represent the boundary geometries), $U=\sum_b u_b b$. To read off coefficients from equalities of two of these functionals (SDE's arise from such relations) one introduces the *graph calculus* [Pér18a]. Algebraically, this is modelled on the monoid algebra $\mathcal{A}[\mathcal{G}]$ corresponding to a certain function space $\mathcal{A}$ and to the free monoid $\mathcal{G}$ spanned by graphs. 'Graph derivatives' $\partial b/\partial b\neq 1$, but:

$$\frac{\partial b}{\partial c}=\delta(b,c)\times\text{ orbit of }\mathrm{Aut_c}(b)\text{ on the function space }\mathcal{A}.$$

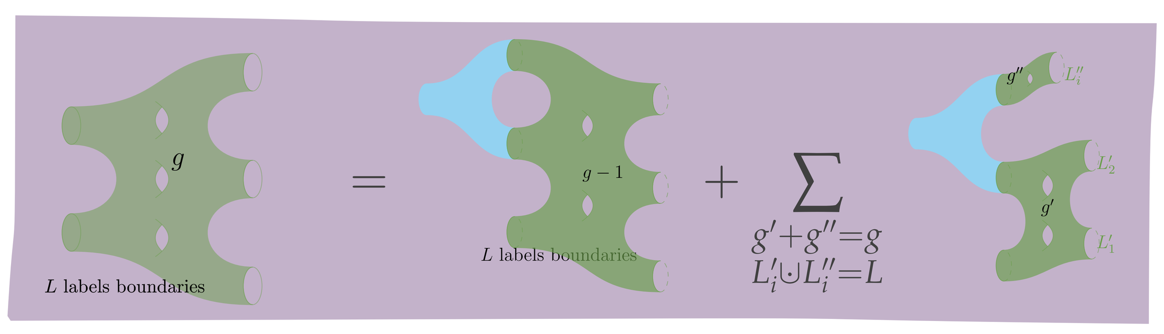
  If b is disconnected, $b=g_1^{\alpha_1}\cdots g_n^{\alpha_n}$, and the g_i's are connected, pairwise non-isomorphic, $\mathrm{Aut_c}(b)=\prod_a\mathrm{Aut_c}(g_a)\wr\mathfrak{S}(\alpha_a)$. The corresponding Leibniz rule takes this into account and yields SDE's for connected correlators of disconnected boundary.

- *Large-N Schwinger-Dyson Equations* (joint work with R. Pascalie, A. Tanasă and R. Wulkenhaar). Gurău's large-N limit [Gur12] generalized to Tensor Field Theory. Closed equation in the LO of the large-N limit for the 2-point function G, [PPTW19]

$$G(\mathbf{x})=\Big(|\mathbf{x}|^2+2\lambda\sum_{a=1}^{3}\int d\mathbf{q}_{\hat a}G(\mathbf{q}_{\hat a}x_a)\Big)^{-1}.$$

Outlook

- Obtain these SDE's for each fixed ω sector of $G_{\mathcal{D}}$
- Causality of Tensor Models and Tensor Field Theory
- Gauge interactions on random discrete spaces (random colored graphs)
- Solution of the 2-point equations and recursions
- Generalization of Eynard-Orantin-Chekhov Topological Recursion



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