Super Quantum Airy Structures

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Motivation: 2d Quantum Gravity

Edward Witten conjectured in [1] that two models of 2d quantum gravity are equivalent. One of the models is given by the matrix integral:

$$Z_1(t_\star) = \int_{\mathcal{H}_N} \exp\Big(\sum_{i=0}^{\infty} t_i \mathrm{Tr}(X^i)\Big)\mathrm{d}X,$$

where the integration is performed over the set $\mathcal{H}_N$ of $N \times N$ Hermitian matrices. This integral is an element of the space of formal power series in the infinite number of variables $\mathbb{C}[[t_0, t_1, \dots]]$. The relation to the quantum gravity comes from the fact that Z_1 can be expressed as sum over graphs, which can thought of as duals to the random geometries. The second model is given by a different series:

$$Z_2(x_\star) = \exp\Big(\sum_{k \in \mathbb{Z}^{\mathbb{N}}} \langle \tau_0^{k_0}\tau_1^{k_1}\dots\rangle \prod_i \frac{x_i^{k_i}}{k_i!}\Big),$$

where

$$\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \hbar^{g-1}\int_{[\bar{\mathcal{M}}_{g,n}]^{vir}} \prod_{i=1}^{n} \psi_i^{d_i}, \qquad (1)$$

and $\bar{\mathcal{M}}_{g,n}$ be the moduli space of stable Riemann surfaces of genus g and n marked poitns. The cohomology classes ψ_i are first Chern classes of the blundles $\mathcal{L}_i$ defined as follows. Fibre of this bundle at any curve C is the fibre of the cotangent bundle of C at i^{th} marked point: $\mathcal{L}_i|_C = T^*C|_{p_i}$. The cycyle over which the integral is performed $[\bar{\mathcal{M}}_{g,n}]^{vir}$ is the virtual fundamental class [2].
In his Ph.D. thesis Maxim Kontsevich proved that those two models, under certain identification of variables, are equivalent [3]. The consequence of this fact is an infinite family of equations $L_i Z_2(x_\star) = 0$ for $i \geq 0$, where [4]:

$$\begin{aligned}
L_0 &= \hbar\frac{\partial}{\partial x_0} - \frac{1}{2}x_0^2 - \sum_{k=0}^{\infty}\hbar x_k \frac{\partial}{\partial x_{k+1}},\\
L_1 &= \hbar\frac{\partial}{\partial x_1} - \hbar\sum_{k=0}^{\infty}\frac{2k+1}{3}x_k\frac{\partial}{\partial x_k} - \frac{1}{24}\hbar, \qquad (2)\\
L_i &= \hbar\frac{\partial}{\partial x_i} - \hbar\sum_{k=0}^{\infty}\frac{(2i+2k-1)!!(2k+1)}{(2i+1)!!(2j+1)!!}x_k\frac{\partial}{\partial x_{i+k-1}}\\
&\quad - \frac{1}{2}\hbar^2\sum_{k=0}^{i-2}\frac{(i-k-2)!!(2k+1)!!}{(2k+1)!!}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_{i-2-k}} \quad \text{for } i \geq 2.
\end{aligned}$$

Moreover, the operators L_n satisfy $\hbar$-deformed Virasoro algebra:

$$[L_i, L_j] = \hbar(i-j)L_{i+j}.$$

Quantum Airy Structures

The collection of operators (2) provide an example of a quantum Airy structure, notion introduced in [5, 6].

Definition

A quantum Airy strucutre is a collection of differential operators $\{L_i\}_{i\in I}$ in variables $\{t_i\}_{i\in I}$, which are of the form

$$L_i = \hbar\frac{\partial}{\partial t_i} + Q_i + \hbar d_i$$

with Q_i purely quadratic in t_i's and $\hbar\frac{\partial}{\partial t_i}$'s, $d_i \in \mathbb{C}$, and such that

$$[L_i, L_j] = \hbar\sum_{k\in I} f_{ij}^k L_k.$$

for some $f_{ij}^k \in \mathbb{C}$ and a formal parameter $\hbar$.

Quantum Airy strucutres were introduced as a reformulation and generalisation of the topological recursion [7]. The basic result concerning quantum Airy structures is the following [5, 6]:

Theorem

Let $\{L_i\}_{i\in I}$ be a quantum Airy structures in variables $\{t_i\}_{i\in I}$. Then there exists a unique collection of symmetric tensors $F_{g,n}(i_1,\dots,i_n)$ such that

$$Z(t_\star) = \exp\Big(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{g-1}}{n!}\sum_{i_1,\dots,i_n\in I}F_{g,n}(i_1,\dots,i_n)t_{i_1}\cdots t_{i_n}\Big)$$

satisfies $L_i Z(t_\star) = 0$ for $i \in I$.

It follows that all the intersection numbers $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}$ are uniquely specified by the equations (2).

Super Quantum Airy Structures

Super quantum Airy structures [8] are an extension of the previous definition including fermionic degrees of freedom. In our case those are formal variables $\{s_j\}_{j\in J}$, which are anticommuting: $s_a s_b = -s_b s_a$. Formal power series in those variables are therefore elements of the exterior algebra $\Lambda^\bullet V$, where $V = \mathrm{lin}\{s_j\}_{j\in J}$. Taking into account both types gives us graded-symmetric variables:

$$x_i x_j = (-1)^{|x_i||x_j|}x_j x_i,$$

where $|x_i| \in \mathbb{Z}_2$ is the grading, equal 0 for bosonic and 1 for fermionic variables.

Definition

A super quantum Airy strucutre is a collection of differential operators $\{L_i\}_{i\in I}$ in $\mathbb{Z}_2$-graded variables $\{x_i\}_{i\in I}$, which are of the form $L_i = \hbar\frac{\partial}{\partial x_i} + Q_i + \hbar d_i$ with Q_i purely quadratic, $d_i \in \mathbb{C}$, grading of L_i equals the grading of x_i and such that

$$[L_i, L_j]_g = \hbar\sum_{k\in I} f_{ij}^k L_k.$$

for some $f_{ij}^k \in \mathbb{C}$. Here $[L_i, L_j]_g = L_iL_j - (-1)^{|L_i||L_j|}L_jL_i$ is the graded commutator.

The existence and uniqueness theorem holds also in the graded case [8]:

Theorem

Let $\{L_i\}_{i\in I}$ be a super quantum Airy structures in variables $\{x_i\}_{i\in I}$. Then there exists a unique collection of graded-symmetric tensors $F_{g,n}(i_1,\dots,i_n)$ such that

$$Z(x_\star) = \exp\Big(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{g-1}}{n!}\sum_{i_1,\dots,i_n\in I}F_{g,n}(i_1,\dots,i_n)x_{i_1}\cdots x_{i_n}\Big)$$

satisfies $L_i Z(x_\star) = 0$ for $i \in I$.

Example

Making link with (2) we will give an example based on superalgebra being an extension of the Virasoro algebra. Such an algebra satisfies the following graded commutation relations ($i \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$):

$$\begin{aligned}
&[\,L_i, L_j]_g = \hbar(i-j)L_{i+j}, \quad [L_i, G_r]_g = \hbar(\frac{1}{2}i - r)G_{i+r},\\
&[\,G_r, G_q]_g = 2\hbar L_{r+q},
\end{aligned}$$

with gradings $|L_i| = 0$ and $|G_r| = 1$. Explicilty the operators are given by [8]:

$$\begin{aligned}
L_i &= \hbar\frac{\partial}{\partial x_i} + \frac{\hbar^2}{2}\sum_{k=0}^{i}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_{i-k}} + \hbar\sum_{k=i+1}^{\infty}(k-i)x_{k-i}\frac{\partial}{\partial x_k} + \frac{\hbar}{2}\sum_{r=\frac{1}{2}}^{\infty}(r+\frac{i}{2})\theta_{r+\frac{1}{2}}\frac{\partial}{\partial\theta_{i+r+\frac{1}{2}}}\\
&\quad + \frac{1}{2}\hbar^2\sum_{r=\frac{1}{2}}^{i-\frac{1}{2}}\frac{\partial}{\partial\theta_{r+\frac{1}{2}}}\frac{\partial}{\partial\theta_{i-r+\frac{1}{2}}} + \frac{1}{2}\hbar\sum_{r=i+\frac{1}{2}}^{\infty}(-r+\frac{i}{2})\theta_{r-i+\frac{1}{2}}\frac{\partial}{\partial\theta_{r+\frac{1}{2}}} + \hbar D\delta_{i=0},\\
G_r &= \hbar\frac{\partial}{\partial\theta_{r+\frac{1}{2}}} + \frac{1}{2}\hbar^2\sum_{m=0}^{r-\frac{1}{2}}\frac{\partial}{\partial\theta_{r+\frac{1}{2}-m}}\frac{\partial}{\partial x_m} + \frac{1}{2}\hbar\sum_{m=1}^{\infty}x_m\frac{\partial}{\partial\theta_{r+\frac{1}{2}+m}}\\
&\quad + \frac{1}{2}\hbar\sum_{m=r+\frac{1}{2}}^{\infty}\theta_{m-r+\frac{1}{2}}\frac{\partial}{\partial x_m},
\end{aligned}$$

where gradings are $|x_i| = 0$ and $|\theta_r| = 1$ and $D \in \mathbb{C}$ is a parameter.
Finding interesting applications of the super quantum Airy structures is an open problem. One possibility is in computing intersection numbers similar to (1), where however odd cohomology classes are present.

References

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