Charges of conformal symmetries and their applications in plane gravitational waves

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Abstract

Focusing on the maximal conformal symmetry of vacuum solutions to the Einstein equations, conserved charges associated with conformal generators are discussed. The meaning of these charges is given by means of the Lewis invariants which subsequently are used to solve explicitly equations of motion for a particle in some plane gravitational waves.

Introduction

The analysis of the motion of a particle in gravitational fields is, in general, very complicated. Only for backgrounds exhibiting rich symmetries it can be simplified, due to integrals of motion associated with symmetry generators. For the gravitational fields the symmetry is usually understood as the isometry group (equivalently, the algebra of Killing vectors). With any Killing vector one can associate an integral of motion. When K is a conformal vector field with the conformal factor 2ψ then along the affine parameterized geodesic the identity holds $\frac{dI_K}{d\tau} = m\psi$. For example, if K is a homothetic field then $\psi = \psi_0 = const$ and $I_K - m\tau\psi_0$ is an integral of motion. Of course, in general, ψ cannot be put directly under the derivative sign; in consequence one obtains a non-local integral of motion. However, there are some special cases where this procedure is possible and can be useful.

Main results

In the case of, non-flat, vacuum solutions to Einstein's equations the maximal conformal symmetry group is 7-dimensional. The maximal dimension is realized by three classes of metrics; all of them belong to the plane gravitational waves (PGW)

$$g \equiv \mathbf{x} \cdot H(u)\mathbf{x}du^2 + 2dudv + d\mathbf{x}^2. \tag{1}$$

For the first and second classes the isometry group is 6-dimensional and there is a homothety $V = 2v\partial_v + \mathbf{x}\cdot\nabla$. Since for PGW the generic dimension of the isometry group is 5, both classes exhibit the maximal dimension of the isometry group, and they were extensively studied. Moreover, both of them do not admit the proper conformal transformation. The third class consists of two families [1]: the first family, forming linearly polarized waves, is defined by

$$H^{(1)}(u) = \frac{a}{(u^2+\epsilon^2)^2}G^{(1)}(u), \qquad G^{(1)}(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and the second, circularly polarized, one by the profiles

$$H^{(2)}(u) = \frac{a}{(u^2+\epsilon^2)^2}G^{(2)}(u), \quad G^{(2)}(u) = \begin{pmatrix} \cos(\phi(u)) & \sin(\phi(u)) \\ \sin(\phi(u)) & -\cos(\phi(u)) \end{pmatrix}, \tag{3}$$

where $\phi(u) = \frac{2\gamma}{\epsilon}\tan^{-1}(u/\epsilon)$ and $\gamma > 0$.

Besides the 5-dimensional isometry group and the mentioned homothety there is a proper conformal vector [1]. In the linearly polarized case it reads

$$K^{(1)} = (u^2+\epsilon^2)\partial_u - \frac{1}{2}\mathbf{x}^2\partial_v + u\mathbf{x}\cdot\nabla, \quad \psi = u. \tag{4}$$

The corresponding integral of motion takes the form

$$I^{(1)} = -\frac{m^2\epsilon^2}{2p_v^2} + a\frac{(x^1)^2-(x^2)^2}{2(u^2+\epsilon^2)} - (u^2+\epsilon^2)\frac{\dot{\mathbf{x}}^2}{2} - \frac{1}{2}\mathbf{x}^2 + u\mathbf{x}\cdot\dot{\mathbf{x}}. \tag{5}$$

Now, the key observation is that $I^{(1)}$ can be rewritten in terms of the Lewis invariants [2]:

$$I^{(1)} = -\frac{m^2\epsilon^2}{2p_v^2} - \frac{\epsilon}{2}\left[(\rho\dot{x}^1 - \dot{\rho}x^1)^2 + \frac{\Lambda_1(x^1)^2}{\rho^2}\right] - \frac{\epsilon}{2}\left[(\rho\dot{x}^2 - \dot{\rho}x^2)^2 + \frac{\Lambda_2(x^2)^2}{\rho^2}\right], \tag{6}$$

where Λ_i are some constants and the function $\rho(u) = \sqrt{u^2+\epsilon^2}/\sqrt{\epsilon}$ satisfies the set of the Milne-Pinney equations. Thus, according to the general procedure [3], the transformation

$$\frac{d\tilde{u}}{du} = \frac{1}{\rho^2(u)}, \quad \mathbf{x} = \sqrt{\epsilon}\rho(u)\tilde{\mathbf{x}}, \tag{7}$$

relates the u-dependent linear oscillator, defined by H, to another one. In our case ρ yields the so-called Niederer transformation [4]

$$u = \epsilon\tan(\tilde{u}), \quad \mathbf{x} = \frac{\epsilon\tilde{\mathbf{x}}}{\cos(\tilde{u})}, \tag{8}$$

which is known from the fact that it relates, locally, the free motion to the half of period harmonic motion. Thus for $g^{(1)}$ the transverse part of the geodesic equations, in the new variables, takes the form of the harmonic oscillator and consequently can be easily solved. Now, for the second, circularly polarized, family of PGW the conformal field $K^{(2)}$ is of the form

$$K^{(2)} = K^{(1)} - \gamma(x^2\partial_1 - x^1\partial_2), \psi = u. \tag{9}$$

Then the corresponding integral of motion takes the form

$$I^{(2)} = -\frac{m^2\epsilon^2}{2p_v^2} - \frac{\epsilon}{2}\left[(\rho\dot{\mathbf{x}} - \mathbf{x}\dot{\rho})^2 - \frac{\mathbf{x}}{\rho}\cdot(\frac{G^{(2)}(u)}{\epsilon^2} - I)\frac{\mathbf{x}}{\rho}\right] - \gamma\dot{\mathbf{x}}\times\mathbf{x}. \tag{10}$$

Let us note that the terms in the square brackets do not form the ordinary Lewis invariants ($G^{(2)}$ depends explicitly on u). However, the function ρ satisfies Milne-Pinney type equation with $H^{(2)}$ and a suitable symmetric matrix $\tilde{H}^{(2)}$ (the explicit form is given in [5]). Thus by means of the Niederer transformation the transverse part of geodesic equations takes the form

$$\tilde{\mathbf{x}}'' = \tilde{H}^{(2)}(\tilde{u})\tilde{\mathbf{x}}. \tag{11}$$

In contrast to the previous case, this time the new linear oscillator is again time-dependent. Despite this the explicit solution can be obtained. Indeed, in the new coordinates $\mathbf{y}$

$$\tilde{\mathbf{x}} = R(\tilde{u})\mathbf{y}, \tag{12}$$

where $R(\tilde{u})$ is a rotation with the angular frequency $\omega = \frac{\gamma}{\epsilon}$, the geodesic equations take the following form

$$\begin{aligned} (y^2)'' + 2\omega(y^1)' + \Omega_- y^2 = 0, \\ (y^1)'' - 2\omega(y^2)' + \Omega_+ y^1 = 0, \end{aligned} \tag{13}$$

where

$$\Omega_\pm = 1 - \omega^2 \mp \Omega, \qquad \Omega = \frac{a}{\epsilon^2}. \tag{14}$$

The above set of equations can be explicitly solved and consequently the initial ones also. The solutions obtained enable more explicit analysis of the interaction of a particle with the plane gravitational pulses.

Furthermore, the charges, can be interpreted in the new coordinates as the "classical" energy.

$$I^{(1)} = -\epsilon^2\left[\frac{m^2}{2p_v^2} + E^{(1)}\right], \tag{15}$$

where

$$E^{(1)} = \frac{1}{2}\tilde{\mathbf{x}}'^2 - \frac{1}{2}\tilde{\mathbf{x}}\cdot\tilde{H}^{(1)}\tilde{\mathbf{x}} = \frac{1}{2}\tilde{\mathbf{x}}'^2 + \frac{\Lambda_1(\tilde{x}^1)^2}{2} + \frac{\Lambda_2(\tilde{x}^2)^2}{2}. \tag{16}$$

Similarly, for the second family and $y's$ coordinates; $I^{(2)}$ is related to total energy for the system defined by the equations (13)

$$I^{(2)} = -\epsilon^2\left[\frac{m^2}{2p_v^2} + E^{(2)}\right], \tag{17}$$

where

$$E^{(2)} = \frac{1}{2}(\mathbf{y}')^2 + \frac{1}{2}\Omega_+(y^1)^2 + \frac{1}{2}\Omega_-(y^2)^2, \tag{18}$$

and $\Omega_\pm$ are given by eqs. (14).

To conclude let us note that taking $a = \frac{2\epsilon^3}{\pi}$ and $\gamma = r\epsilon$, i.e. $\omega = r > 0$, in the limit $\epsilon \mapsto 0$ one obtains the Dirac delta profile

$$H^{(2)}(u) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(u)\cdot\begin{cases} \frac{\sin(\pi r)}{\pi r(1-r^2)}, & r \neq 1; \\ \frac{1}{2}, & r = 1. \end{cases} \tag{19}$$

In contrast to sandwiches approach, the contraction $\epsilon \mapsto 0$ of the conformal algebra yields a conformal algebra for the Dirac delta profile. For more details and applications see [5].

Final remarks

It turns out that the discussed conformal symmetry preserves also some electromagnetic fields, thus the above results can be directly extend to some additional electromagnetic backgrounds as well as fit into recent investigations of non-local charges in [6] and classical double copy idea [7].

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